Semiclassical Description of Chaos-Assisted Tunneling

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We study tunneling between regular and chaotic regions in the phase space of Hamiltonian systems. We analytically calculate the transition rate and show that its variation depends only on corresponding phase space area and in this sense is universal. We derive the distribution of level splittings associated with the pairs of quasidegenerate regular eigenstates which in the general case is different from a Cauchy distribution. We show that chaos-assisted tunneling leads to level repulsion between regular eigenstates, solving the longstanding problem of level-spacing distribution in mixed systems.

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Quantum-mechanical tunneling in systems whose classical counterparts show a mixture of regular and chaotic dynamics, has been a topic of great interest in the past decade [1–10]. In these systems, which represent the majority of dynamical systems found in nature, tunneling between regular and chaotic regions in the phase space leads to anomalously high transition rates and often dominates system observables, such as, e.g., atom ionization energies [11] and optical resonance lifetimes [6]. Of particular interest is the case of tunneling between regular regions in the phase space separated by a chaotic layer. Contrary to the intuition based on textbook examples of tunneling in integrable systems, the corresponding tunneling rate can be orders of magnitude higher than what would be expected from direct processes [3–6,8,10]. Instead, the transition rate is determined by chaos-assisted tunneling (CAT) which can be visualized as a sequence of tunneling to the chaotic layer, followed by the *classical* propagation until the chaotic diffusion brings the particle to the neighborhood of the other regular region, followed by the last ''short distance tunneling'' to the destination. Even though in a formal perturbation theory CAT is a higher-order process compared to direct transitions from one regular region to the other, chaos-assisted contribution is orders of magnitude higher [3–6,8,10] since a large part of the phase space separating the initial and final states is traversed via a classically allowed process.

However, all existing treatments of CAT [2–4,11–15] suffer from one major problem—they generally consider the average coupling between regular and chaotic eigenstates as a fitting parameter. The only exception to this so far is the approach of Ref. [5], which used special properties of the scattering matrix of circular cylinder to evaluate level splittings in annular billiard, and thus could not be extended to other systems. To develop a general and quantitative description of CAT it is therefore necessary to address the problem of the matrix elements between regular and chaotic eigenstates.

Also, the existing treatments of CAT often concentrate on the systems with a discrete spatial symmetry [4,11]. For example, the Cauchy distribution of level splittings due to chaos-assisted tunneling derived in Refs. [4,11], applies only to symmetric systems. However, as it has been shown by Tomsovic [16], the origin of CAT-induced doublets lies in the time-reversal symmetry and CAT determines the doublet splitting even in the absence of spatial symmetry.

In the current Letter we develop a theoretical description of chaos-assisted tunneling which is free from above-mentioned limitations. Using semiclassical methods, we calculate the matrix element for coupling between regular and chaotic states, and demonstrate that the variation of the tunneling rate only depends on the corresponding phase space areas and the effective Plank's constant. In this sense, the transition rate due to chaos-assisted processes is universal in all dynamical systems. Furthermore, in the experimentally relevant parameter range the chaos-assisted tunneling rate is not simple exponential, as it has been suggested earlier [10,12,13]. We also study the distribution of level splittings due to chaos-assisted tunneling and derive the general form of level splitting distribution. We show that when no exact spatial symmetry is present the resulting distribution is different from Cauchy distribution due to a strong suppression of small splittings. Finally, we address the long-standing problem of level-spacing distribution in mixed systems [17], and demonstrate that CAT-induced level repulsion is responsible for deviations from celebrated Berry-Robnik formula [18].

The Hamiltonian of a system with regular-chaotic dynamics can be written in the following form:

$$\hat{H} = \sum_R E_R |\psi_R\rangle\langle\psi_R| + \sum_C E_C |\psi_C\rangle\langle\psi_C| + \sum_{RC} \{V_{RC} |\psi_R\rangle\langle\psi_C| + \text{c.c.}\}, \qquad (1)$$

where E_R and ψ_R are the energies and wave functions of the regular states, E_C, ψ_C are the energies and wave functions of chaotic states, and V_{RC} describe the interaction between the regular and chaotic states. Interaction matrix elements V_{RC} strongly fluctuate and can be adequately described by an (uncorrelated) Gaussian distribution [4,11,12].

Regular objects in the phase space generally include the remaining KAM tori and stability islands. In the present Letter, we concentrate on the case of hard chaos when the remaining KAM tori are few, and islands of stability dominate the regular portion of the phase space. The extension of our treatment to the tori will be presented elsewhere [19].

In order to calculate the variance of interaction matrix element we consider our system to be a perturbation of a regular system with surface of section consisting of "tile" of regular islands (Fig. 1, left panel). A straightforward semiclassical quantization [20] of the classical motion within each of these islands yields a set of wave functions localized at the islands $\Psi_{mn}(q - q_\alpha, p_\alpha)$ where m and n are the integers, α is the index of the island, and (q_α, p_α) correspond to the coordinate and momentum of the center of the island in the surface of section (analytical expressions for Ψ_{mn} can be found in Ref. [20]).

Next, we introduce a nonintegrable perturbation to this tile system such that all but the central island ($\alpha = 0$) of the tile are destroyed. The regular states which are still localized at the surviving island, can be adequately described by the functions $\Psi_{mn}(q - q_0, p_0)$. On the other hand, the destruction of all the other islands and the formation of the chaotic sea (see Fig. 1, right panel) implies that the chaotic wave functions can be represented as a superposition of the island wave functions of $\alpha \neq 0$ with quasirandom coefficients.

Using a technique similar to Bardeen's tunneling Hamiltonian formalism [21] and taking advantage of the known analytical forms for the functions $|m, n, \alpha\rangle$ and their overlaps $\langle m, n, \alpha | m', n', \beta\rangle$, we arrive at the following expression for the variance of the "ground state" ($m = 0$) interaction matrix element:

$$V^2 \equiv \langle |V_{RC}|^2\rangle = c_0 \hbar_{\rm eff}^2 \frac{\Gamma(\frac{A}{\pi\hbar_{\rm eff}}, \frac{2A}{\pi\hbar_{\rm eff}})}{\Gamma(\frac{A}{\pi\hbar_{\rm eff}} + 1, 0)}, \qquad (2)$$

where A is the area of the regular island in the phase space, $\hbar_{\rm eff}$ is effective (dimensionless) Plank's constant, Γ is incomplete Gamma function [22], and c_0 is a nonuniversal prefactor which does not depend on $\hbar_{\rm eff}$ and can be calculated for each particular system.

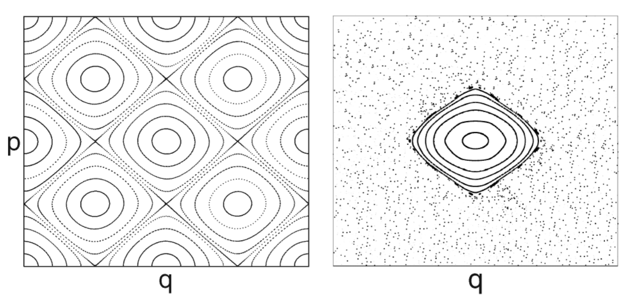

FIG. 1. Surface of section of unperturbed (regular) system (left) and of the perturbed system with mixed regular chaotic dynamics (right).

If the energies of the two symmetry-related regular states are degenerate, CAT removes this degeneracy. CAT-induced splitting between the regular levels Δ can be calculated using standard perturbation approach for the degenerate levels [23], leading to the following expression:

$$\Delta = \sqrt{\left(\sum_C \frac{|V_{1C}|^2 - |V_{2C}|^2}{E_R - E_C}\right)^2 + \left|\sum_C \frac{2V_{1C}V_{2C}^*}{E_R - E_C}\right|^2}. \qquad (3)$$

Note that in the perturbative description leading to Eq. (3) the mean level splitting is characterized by a single energy scale $\nu = (V^2/D)$, where $D = \langle |E_R - E_C|\rangle$ is a mean level spacing. Since (in two-dimensional systems) $D \propto \hbar_{\rm eff}$ [24], the typical CAT level splitting is given by

$$\Delta \propto \hbar_{\rm eff} \frac{\Gamma(\frac{A}{\pi\hbar_{\rm eff}}, \frac{2A}{\pi\hbar_{\rm eff}})}{\Gamma(\frac{A}{\pi\hbar_{\rm eff}} + 1, 0)}. \qquad (4)$$

As follows from Eqs. (2) and (4), the variation of the transition rate of CAT and the corresponding level splittings depend only on the effective Plank's constant $\hbar_{\rm eff}$ and the area of the regular island in the phase space A. In this sense, the CAT rate is universal.

To demonstrate the accuracy of our results and their universality, we compare our theory to numerical data for two very different physical systems: (i) a deformed cylinder optical resonator, such as the one used in novel microdisk lasers [7], and (ii) system of cold atoms trapped by periodic laser beams [12].

First, we consider the example of the splittings of the bow-tie wave functions in a deformed quadrupole resonator, with shape described by

$$R(\phi) \propto 1 + 0.15\cos(2\phi) + \delta r(\phi), \qquad (5)$$

where δr corresponds to short-range roughness represented by high-order harmonics, with $\delta r \ll 1$.

These bow-tie wave functions supported by regular islands in the surface of section [see Fig. 2(a)], correspond to the lasing modes in the novel noncircular microdisk lasers [7]. In Fig. 2(b) we plot bow-tie mode splittings vs the wave number (which defines the effective Plank's constant in this system $\hbar_{\rm eff} = 1/kR$). Note that while the mode splittings change by more than 2 orders of magnitude, our analytical formula (4) yields an agreement with the numerical calculations *with average error of* $\simeq 1\%$.

Doublets of eignestates supported by regular regions separated in the phase space by chaotic layer, are also formed in the system of cold atoms trapped by periodic laser beams [12]. The corresponding eignestates are localized at two regular islands, indicated by the arrows in Fig. 2(c). In Fig. 2(d) we plot these level splittings taken from Ref. [12] (these data were not averaged, thus leading to strong oscillations which are also a signature of chaos-assisted tunneling [4]), and compare them to our result

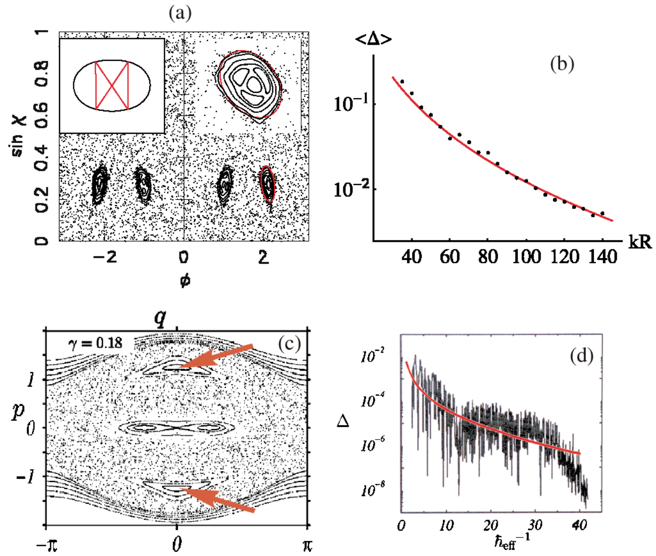

FIG. 2 (color). (a) The surface of section (SOS) for the distorted quadrupole billiard (main picture, upper half of SOS is shown), the shape of a billiard with bow-tie orbit shown (left inset) and the enlarged bow-tie island (right inset). The area of the red ellipse corresponds to the area used to calculate the analytical curve in panel (b); (b) Comparison between the mean level splitting the bow-tie modes in distorted quadrupole billiard (dots) and Eq. (4) (red line) (note that in optical resonators [10] $\hbar_{\rm eff} = 1/kR$); (c) SOS for the system of cold atoms in periodic laser field [12]. Arrows show the position of the symmetry-related regular states; (d) Comparison between the CAT-splittings between symmetry-related states in the system in (c) [12] (black lines) and our theory (red line).

(4). Note that Eq. (4) adequately describes the appearance of the "plateau" [12] for $\hbar_{\rm eff}^{-1} > 10$ (see Fig. 2(d)), which is simply due to the fact that, as follows from Eq. (2), even for $\hbar_{\rm eff}^{-1}$ as large as 100 the dependence of the mean splitting is not a simple exponential.

We now consider the distribution of the energy level splittings. If the chaotic spectrum has the same symmetry as the regular states, then the regular energy level splitting is given by Cauchy statistics [3,4,12]

$$P(\Delta) = \frac{1}{\sqrt{\pi}} \frac{4\nu}{4\pi\nu^2 + \Delta^2} \qquad (6)$$

[see Fig. 3(b) and the discussion below]. In the following calculations we suppose that the chaotic spectrum does not possess such a symmetry.

Introducing the dimensionless interaction matrix element, $v_{ij} = V_{ij}/V$, level-spacing $d_C = (E_R - E_C)/D$, and the level splitting $x = \Delta/\nu$ we find that the level splitting distribution is then given by

$$p(\Delta) = \int \cdots \int \delta(\Delta - \nu x) p(\mathbf{v}_1, \mathbf{v}_2, \mathbf{d}) d\mathbf{v}_1 d\mathbf{v}_2 d\mathbf{d}, \qquad (7)$$

where $p(\mathbf{v_1}, \mathbf{v_2}, \mathbf{d})$ is the joint probability of the interaction matrix elements $\mathbf{v_1} \equiv (v_{11}, \ldots, v_{1N})$, $\mathbf{v_2} \equiv (v_{21}, \ldots, v_{2N})$, and the corresponding energy level spacings $\mathbf{d} \equiv (d_1, \ldots, d_N)$.

The chaotic part of the mixed system in the presence of time-reversal symmetry is well described by a Gaussian orthogonal ensemble (GOE) of random matrices [24]. Following [4] we consider the chaotic part of the energy spectrum to be equidistant and V_{ij} to be Gaussian random variables. Straightforward, though cumbersome calculations show that the following function adequately describes the distribution of the level splitting for GOE case

$$p(\Delta) = \frac{\Delta}{\nu^2}\left[F\left(\frac{\Delta}{\nu}, 1\right) + 3F\left(\frac{\Delta}{\nu}, -\frac{1}{3}\right)\right], \qquad (8)$$

$$F(x, q) = \frac{1}{16\pi^3}\int_0^1 dy \frac{\sqrt{y}\exp\left(-\frac{yx^2}{8\pi^2(1+q\sqrt{6-4y})}\right)}{\sqrt{1-y}(1 + q\sqrt{6-4y})}.$$

Note that in the limit of large splittings the obtained distribution is consistent with the derived earlier Cauchy behavior [4], however in the limit of small splitting it predicts linear behavior due to the level repulsion in the chaotic spectrum.

To verify the accuracy of approximation used to derive Eq. (8), we compare our distribution to a numerical simulation where the chaotic spectrum was generated by diagonalization of a Gaussian random matrix ensemble. Figure 3(a) illustrates the excellent agreement between Eq. (8) and the numerical simulations.

We also compare our theoretical results to the numerical spectra corresponding to a physical system — a cylinder with distorted quadrupolar deformation in the cross section, described by Eq. (5). We generate the ensemble of resonators with fixed bow-tie island structure and different shape perturbations δr (such shape variation corresponds to the changes in chaotic modes of the system), and compare the level splitting distribution for the bow-tie modes to the analytical formula (8) and to Cauchy distribution (6) for a fixed value of kR. The symmetry of the chaotic mode spectra is governed by

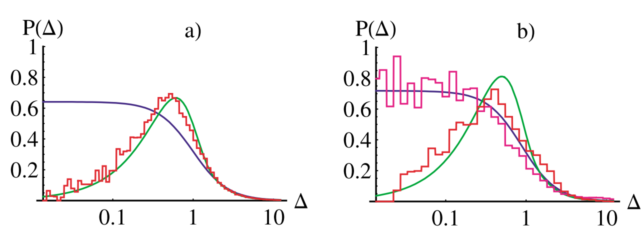

FIG. 3 (color). (a) Comparison between the level splitting (normalized to the median [5]) distribution for random matrix simulated data (red curve), Eq. (8) (green curve), and Cauchy distribution (blue curve). (b) Comparison between the level splitting distribution for the bow-tie orbit in quadrupole billiard for $kR = 50$ with symmetric (pink curve) and asymmetric (red curve) distortion δr, analytical prediction given by Eq. (8) (green curve) and Cauchy distribution given by Eq. (6) (blue curve).

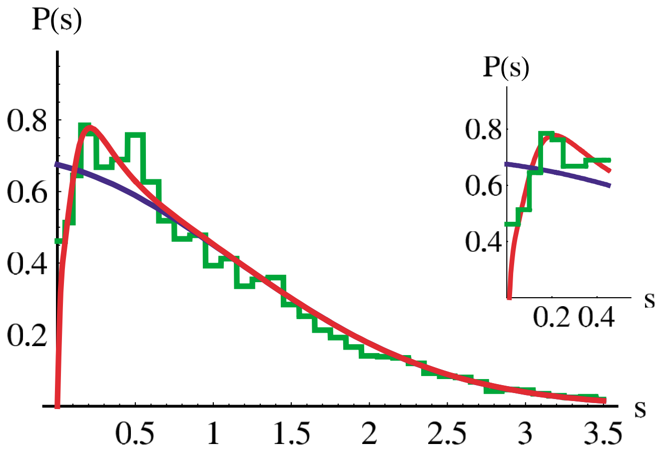

FIG. 4 (color). Comparison between the nearest neighbor spacing distribution in the deformed elliptical microresonator for $kR \simeq 75$ (green), best fit Berry-Robnik distribution (blue) and our analytical result (red); the inset shows the limit of small spacing in more detail.

the symmetry of the "distortion" δr. When the chaotic eigenstates possess quadrupole symmetry, the regular levels couple to the chaotic ones with the same symmetry, leading to the Cauchy distribution of the level splittings — see Fig. 3(b). The situation changes dramatically when the resonator boundary, and therefore the resulting chaotic spectrum are not symmetric, which is often the case in the real systems. In this case both regular states couple to the *same* chaotic state, which leads to the strong level repulsion as shown in Fig. 3(b). Our analytical expression is consistent with the actual distribution, although the agreement is not as good as in the case of GOE simulations considered above. This discrepancy has its origin in remaining "dynamically localized" [25] and "scarred" [26] states supported by the chaotic part of the phase space of the system, whose level statistics is not adequately described by random matrices [24].

Finally, we point out that it is this strong level repulsion that describes the discrepancy between the nearest-neighbor energy level-spacing distribution in the systems with mixed regular-chaotic dynamics and the well-known Berry-Robnik (BR) distribution [18]. We attribute this discrepancy to the independent treatment of the regular and chaotic portions of the spectrum in BR approach. We propose that, in a mechanism similar to the one described above, chaos-assisted tunneling leads to the level repulsion between any two regular levels via an "intermediate" chaotic state. The resulting regular level repulsion will lead to the vanishing probability of zero spacings, in agreement with the actual behavior.

Using a perturbative approach along the lines of the present Letter yields an analytical expression for the level-spacing distribution in systems with mixed dynamics. In Fig. 4 we compare this distribution (the corresponding analytical expression is not shown here due to lack of space) with the numerical calculation for the nearest-neighbor eigenenergy spacings in an elliptical microresonator with short-range roughness. As clearly seen from Fig. 4, the distribution taking into account CAT-induced regular level repulsion, is in excellent agreement with numerical data.

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