Delay Time and the Hartman Effect in Quantum Tunneling

Herbert G. Winful
Department of Electrical Engineering and Computer Science, University of Michigan, 1301 Beal Avenue, Ann Arbor, Michigan 48109-2122, USA
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A general relation between the group delay and the dwell time is derived for quantum tunneling. It is shown that the group delay is equal to the dwell time plus a self-interference delay. The Hartman effect in quantum tunneling is explained on the basis of saturation of the integrated probability density (or number of particles) under the barrier.

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How long does it take a particle to tunnel through a potential barrier? This is a question that has occupied physicists for decades [1] and one for which there is still no definitive answer [2]. Indeed the influential review by Hauge and Støvneng lists at least seven different tunneling time definitions of which two, the so-called phase time (group delay) and dwell time, are considered well established [3]. These times remain controversial, however, because in the opaque barrier limit they predict effective tunneling velocities that exceed the vacuum speed of light and may even become unlimited (the "Hartman effect") [4]. Furthermore, it is not clear how these two time definitions, equal in the classical limit, are related under quantum tunneling conditions where any connection between them has been explicitly denied [5]. In recent papers we have argued that these delay times are not propagation delays and should not be linked to a velocity [6–9]. In particular, we have shown that the phase time is proportional to the time averaged stored energy and is a measure of a cavity lifetime. This proportionality between the phase time and the stored energy was used to explain the paradox of the Hartman effect [6] for the specific case of tunneling electromagnetic waves. While the electromagnetic case is analogous to quantum mechanical particle tunneling [10], there are enough differences between the two to warrant a separate examination of the problem of quantum tunneling in its own right. Other than the fact that the Maxwell equations for tunneling photons are Lorentz invariant while the Schrödinger equation for tunneling electrons is not, the two systems also satisfy different dispersion relations. This should affect both qualitatively and quantitatively the detailed nature of the interference process that results in barrier tunneling.

In this Letter we derive a general and explicit relation between the group delay and the dwell time for quantum tunneling, thus unifying these two approaches to a tunneling time. We show that the group delay is equal to the dwell time plus a self-interference delay which depends on the dispersion outside the barrier. We then show that the Hartman effect for tunneling quantum particles can be explained by the saturation of the integrated probability density (or number of particles) under the barrier, which itself is proportional to the group delay. Our results further confirm that there is nothing superluminal in quantum tunneling.

The one-dimensional tunneling configuration considered here is shown in Fig. 1. In the stationary state description a particle of energy E and momentum $\hbar k$ is incident from the left upon a real potential barrier $V(x)$ that occupies the region $0 < x < L$. The energy and momentum are related through $E = \hbar^2 k^2/2m$, where m is the mass of the particle. Whereas a classical particle is totally reflected by this potential barrier when $E < V$, quantum mechanically there is a finite, albeit small, probability that the particle will tunnel through the barrier and end up in the region $x > L$. The probability of this event is measured by the magnitude squared of the barrier transmission coefficient $T = |T|e^{i\phi_t}$. Figure 2 shows the transmission probability for the case of a rectangular potential barrier $V(x) = V_0$. The particle is much more likely to be reflected, with a probability given by the magnitude squared of the reflection coefficient $R = |R|e^{i\phi_r}$. In 1931 Condon first posed the question of the alacrity of the tunneling process [11]. A year later MacColl asserted, based on a wave packet solution of the Schrödinger equation, that there is "no appreciable delay" in the transmission of the packet through the barrier [1]. Hartman later showed that there is a finite delay but that this delay is shorter than the equal time, the time a particle of equal energy would take to traverse the same distance L in the absence of the barrier [4]. The group delay measures the

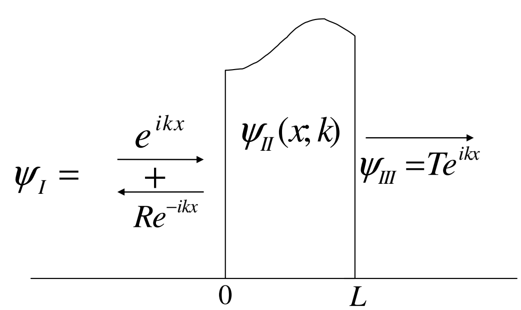

FIG. 1. Schematic of the barrier tunneling problem.

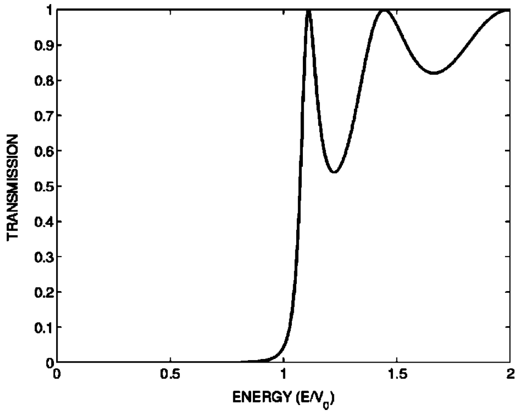

FIG. 2. Transmission of a rectangular potential barrier of height V_0 versus energy E/V_0. Here $\sqrt{2mV_0}L/\hbar \equiv \gamma L = 3\pi$.

delay between the appearance of a wave packet peak at $z = 0$ and its appearance at $z = L$. It is calculated by the method of stationary phase and is given by the energy derivative of the transmission phase shift [12,13]:

$$\tau_{gt} = \hbar d\phi_0/dE, \tag{1}$$

where $\phi_0 = \phi_t + kL$. This is actually the group delay in transmission. The group delay in reflection is given by

$$\tau_{gr} = \hbar d\phi_r/dE, \tag{2}$$

a quantity that differs from the transmission group delay except for symmetric barriers, where $\tau_{gr} = \tau_{gt} \equiv \tau_g$. For a general asymmetric barrier, it is useful to define a bidirectional group delay as the weighted sum of transmission and reflection group delays:

$$\tilde{\tau}_g = |T|^2\tau_{gt} + |R|^2\tau_{gr}. \tag{3}$$

The dwell time is a measure of the time spent by a particle in the barrier region $0 < x < L$ regardless of whether it is ultimately transmitted or reflected. It is given by [5,14]

$$\tau_d = \frac{\int_0^L |\psi(x)|^2 dx}{j_{in}}, \tag{4}$$

where $\psi(x)$ is the stationary state wave function corresponding to energy E and $j_{in} = \hbar k/m$ is the flux of incident particles. We derive a simple and explicit relation between these times and use this relation to explain the origin of the Hartman effect.

We begin with a variational theorem that yields the sensitivity of the wave function to variations in energy. Following Smith [14], we write the time-independent Schrödinger equations for ψ^* and $\partial\psi/\partial E$ and obtain, after some elementary manipulations,

$$\psi^*\psi = \frac{\hbar^2}{2m}\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial E}\frac{\partial\psi^*}{\partial x} - \psi^*\frac{\partial^2\psi}{\partial E\partial x}\right). \tag{5}$$

Upon integration over the length of the barrier we find

$$\left(\frac{\partial\psi}{\partial E}\frac{\partial\psi^*}{\partial x} - \psi^*\frac{\partial^2\psi}{\partial E\partial x}\right)_{x=L} - \left(\frac{\partial\psi}{\partial E}\frac{\partial\psi^*}{\partial x} - \psi^*\frac{\partial^2\psi}{\partial E\partial x}\right)_{x=0} = \frac{2m}{\hbar^2}\int_0^L |\psi|^2 dx. \tag{6}$$

In front of the barrier ($x \leq 0$), the wave function consists of an incident and a reflected component:

$$\psi_I = e^{ikx} + Re^{-ikx}. \tag{7}$$

Behind the barrier ($x \geq L$), there is only a transmitted wave

$$\psi_{III} = Te^{ikx}. \tag{8}$$

By using Eqs. (7) and (8) we evaluate the left-hand side of Eq. (6) as

$$-i2k\left[|T|\frac{d|T|}{dk} + |R|\frac{d|R|}{dk} + i\left(|T|^2\frac{d\phi_0}{dk} + |R|^2\frac{d\phi_r}{dk} + \frac{\mathrm{Im}(R)}{k}\right)\right]\frac{\partial k}{\partial E}.$$

Equating this to the right-hand side in (6), we obtain the bidirectional group delay

$$\tilde{\tau}_g = \frac{\int_0^L |\psi(x)|^2 dx}{j_{in}} - \frac{\mathrm{Im}(R)}{k}\hbar\frac{\partial k}{\partial E}, \tag{9}$$

where we have used the fact that for a lossless barrier $|R|^2 + |T|^2 = 1$. If the barrier is also symmetric, then the bidirectional group delay is the same as the group delay in reflection or in transmission: $\tilde{\tau}_g = \tau_{gt} = \tau_{gr}$.

Equation (9) is a simple and general result that unifies two of the major tunneling times and agrees with the wave packet analysis of Hauge *et al.* [15]. The first term in Eq. (9) is the aforementioned dwell time. The second term is a self-interference term that comes from the overlap of incident and reflected waves in front of the barrier. As the wave packet tunnels through the barrier, part of the incident packet interferes with a portion that has already been reflected [3,15]. This term is of great importance at low energy ($E \to 0$) when the particle spends most of its time dwelling in front of the barrier, interfering with itself, held up in a standing wave, neither coming nor going, its purpose to and fro. Although the role of self-interference in the tunneling process has been recognized, until now the belief had been that its contribution to the group delay could not be disentangled [3,16]. Here we have succeeded in disentangling the role of self-interference and can thus write

$$\tilde{\tau}_g = \tau_d + \tau_i, \qquad (10)$$

where $\tau_i = -\hbar \mathrm{Im}(R) d\ln k/dE$. In quantum tunneling, the approach to the barrier itself involves dispersive propagation and this has an effect on the self-interference process. This was not the case for the tunneling of electromagnetic waves in the photonic band gap structure studied in [6,7] where the approach to the barrier was characterized by nondispersive plane wave propagation. In that case the self-interference delay vanished and the dwell time was shown to equal the group delay.

The self-interference delay can also be written as $\mathrm{Im}(R)/kv$, where $v = \hbar k/m$ is the velocity of the incident particle. In this form we can recognize a connection between this result and the optical theorem for three-dimensional scattering: $\sigma = 4\pi \mathrm{Im} f(0)/k$, where σ is the total scattering cross section and $f(0)$ is the forward scattering amplitude [17]. Both terms arise from interference between incident and scattered waves and lead to a diminution of the forward flux. For the one-dimensional case, the optical theorem can be expressed as $\sigma = 2\mathrm{Im} f(0)/k$ [18]. The backscattering cross section is determined by the imaginary part of the reflection coefficient and the quantity $L_s = \mathrm{Im}(R)/k$ is equivalent to a scattering length. When divided by the incident velocity, this yields a time delay for traversing the scattering length.

The imaginary part of the reflection coefficient, and hence the self-interference delay, can also be related to the Lagrangian for the Schrödinger equation. Consider the function

$$W = \psi^* \nabla \psi \qquad (11)$$

whose imaginary part is proportional to the particle momentum. From the Schrödinger equations for ψ and ψ^* we find that

$$\mathrm{Re} \nabla \cdot (\psi^* \nabla \psi) = |\nabla \psi|^2 + \frac{2m}{\hbar^2}(V - E)|\psi|^2. \qquad (12)$$

For the one-dimensional tunneling problem an integration of Eq. (12) over the barrier region amounts to evaluating W at $x = 0$ and $x = L$ with the use of the wave functions (6) and (7). This calculation yields

$$\frac{\hbar k}{m}\mathrm{Im}(R) = -\frac{1}{\hbar}\int_0^L \left[\frac{\hbar^2}{2m}\left|\frac{d\psi}{dx}\right|^2 + (V - E)|\psi|^2\right] dx. \qquad (13)$$

The integral in Eq. (13) is seen as the Lagrangian for the wave function ψ. The time-independent Schrödinger equation is the Euler-Lagrange equation that results from requiring that the integral in (13) be stationary with respect to path variations that satisfy the boundary conditions. The imaginary part of the barrier reflection coefficient can thus be seen as the result of a stationary action principle. Indeed, for matter waves stationary phase and stationary action are equivalent.

Equation (9) relating the group delay to the dwell time and a self-interference delay is a general relation that holds for any lossless potential. We thus have two completely different methods to calculate the group delay: (i) by calculating the frequency derivative of the transmission phase shift and (ii) by calculating the integrated probability density and the Lagrangian. To test this result, we apply it to the textbook problem of a rectangular barrier in which $V(x) = V_0$, a constant in the region between 0 and L. In this case the transmission phase shift is

$$\phi_0 = -\tan^{-1}(\Delta \tanh \kappa L), \qquad (14)$$

where $\kappa = \sqrt{2m(V_0 - E)}/\hbar$ and $\Delta = (\kappa/k - k/\kappa)/2$. The wave function in the barrier region is

$$\psi = Ce^{-\kappa x} + De^{\kappa x}, \qquad (15)$$

where $C = (1 - ik/\kappa)e^{\kappa L}/2g$, $D = (1 + ik/\kappa)e^{-\kappa L}/2g$, and $g = \cosh \kappa L + i\Delta \sinh \kappa L$. With the use of this wave function Eqs. (9) and (13) yield

$$\tau_d = \frac{mL}{\hbar k}\frac{\cos^2\phi_0}{2}\left[\left(1 + \frac{k^2}{\kappa^2}\right)\frac{\tanh \kappa L}{\kappa L} - \left(\frac{k^2}{\kappa^2} - 1\right)\mathrm{sech}^2 \kappa L\right], \qquad (16)$$

$$\tau_i = \frac{mL}{\hbar k}\frac{\cos^2\phi_0}{2}\left(1 + \frac{\kappa^2}{k^2}\right)\frac{\tanh \kappa L}{\kappa L}, \qquad (17)$$

$$\tau_g = \tau_d + \tau_i = \frac{mL}{\hbar k}\frac{\cos^2\phi_0}{2}\left[\left(\frac{k}{\kappa} + \frac{\kappa}{k}\right)^2 \frac{\tanh \kappa L}{\kappa L} - \left(\frac{k^2}{\kappa^2} - 1\right)\mathrm{sech}^2 \kappa L\right]. \qquad (18)$$

It is readily confirmed that the energy derivative of the transmission phase shift ϕ_0 results in the same expression for the phase time as Eq. (18). It is gratifying that the phase time calculated by two such different methods yields the same result. Figure 3 shows these three times plotted as a function of the normalized energy. For $E < V_0$, there is a significant difference between the phase time and the dwell time because of the self-interference delay. In the classical region, however, these two times become equal.

The Hartman effect is the saturation of the group delay with barrier length. The fact that this time saturates with length indicates that it cannot be a propagation delay and should therefore not be associated with a traversal velocity. In the limit $L \to \infty$, the probability density inside the barrier is simply the decaying exponential $|\psi|^2 \sim \exp(-2\kappa x)$. From (9) it is seen that both the dwell time and the self-interference delay are proportional to the integrated probability density in this exponential limit. This integrated probability density saturates with barrier length and hence the dwell time, self-interference delay, and phase time all saturate. As $L \to \infty$, we find

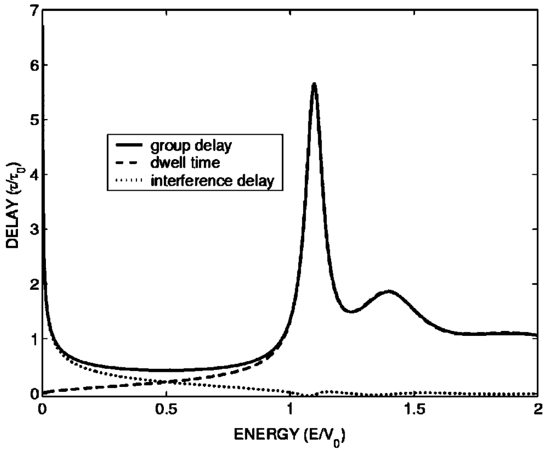

FIG. 3. Dwell time (dashed line), self-interference delay (dotted line), and phase time (solid line) versus normalized energy E/V_0 for a particle in a rectangular potential barrier. Here $\sqrt{2mV_0}L/\hbar \equiv \gamma L = 3\pi$. The times are normalized by $\tau_0 = L/v_0$, where $v_0 = \hbar\gamma/m$.

$$\tau_d = \frac{2}{\kappa v}\left(\frac{E}{V_0}\right), \qquad \tau_i = \frac{2}{\kappa v}\left(\frac{V_0 - E}{V_0}\right),$$
$$\tau_g = \tau_d + \tau_i = \frac{2}{\kappa v}. \tag{19}$$

The times indicated above can be cast in the form of an uncertainty principle by recognizing that $2/\kappa v = \hbar/\sqrt{E(V_0 - E)}$. It is important to note that the delays seen here are not propagation delays and, therefore, their saturation does not imply superluminal and unlimited velocities. In numerical simulations based on the relativistic Klein-Gordon equation, we have shown that the peak of a tunneling wave packet does not even enter the barrier [7,8]. Similar studies with the Dirac equation also show an absence of a wave packet peak in the barrier during tunneling [19]. Thus the peak of the tunneling wave packet does not propagate from input to output. Output and input wave packet peaks are therefore not related by causal propagation and hence the phase time is not a propagation delay. It is a delay associated with the momentary capture and release of a tunneling particle. There is a constant tunneling flux $j_{in}|T|^2$ that arises from the interference between evanescent and antievanescent modes within the barrier. If one divides the integrated particle density in the barrier by this flux, the result is a perfectly "luminal" net-flux delay $\tau_N = \tau_d/|T|^2$ which does not saturate with barrier length [20].

Tunneling without distortion is a narrow band phenomenon requiring wave packets that are narrow in momentum space. The uncertainty principle then demands that the spatial extent of the packet be much greater than the barrier width. As a result, the *duration* of the tunneling event will simply be the temporal extent of the wave packet, assumed propagating at its initial velocity. The shift in the peak of this very broad (in space) wave packet is merely an indication of a scattering phase shift proportional to the integrated probability density under the barrier. Equation (9) also suggests a method for measuring quantities such as the dwell time and the net-flux delay. The group delay can be determined from measurements of scattering phase shifts in transmission or reflection as a function of incident energy. The self-interference delay is obtained from a measurement of the scattering probability and phase shift. A simple subtraction then yields the dwell time. The net-flux delay is then obtained by dividing this dwell time by the transmission coefficient.

In conclusion, we have obtained a general relation between the group delay and the dwell time thus resolving the paradox of the Hartman effect in quantum tunneling.

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