Transition from Kardar-Parisi-Zhang to Tilted Interface Critical Behavior in a Solvable Asymmetric Avalanche Model

A. M. Povolotsky,[1,2] V. B. Priezzhev,[2] and Chin-Kun Hu[1,*]

[1]*Institute of Physics, Academia Sinica, Nankang, Taipei 11529, Taiwan*

[2]*Bogoliubov Laboratory of Theoretical Physics, J.I.N.R., Dubna 141980, Russia*

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We use a discrete-time formulation of the asymmetric avalanche process (ASAP) [Phys. Rev. Lett. **87**, 084301 (2001)] of p particles on a finite ring of N sites to obtain an exact expression for the average avalanche size as a function of toppling probabilities and particle density $\rho = p/N$. By mapping the model onto driven interface problems, we find that the ASAP incorporates the annealed Kardar-Parizi-Zhang and quenched tilted interface dynamics for $\rho < \rho_c$ and $\rho > \rho_c$, respectively, with ρ_c being the critical density for given toppling probabilities and $N \to \infty$. We analyze the crossover between two regimes and show which parameters are relevant near the transition point.

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The properties of driven interfaces have been attracting great attention for many years due to their connection with a variety of physical phenomena such as fluid flow through porous media, motion of charge density waves, flux lines in superconductors, etc. [1–4]. The results obtained can be roughly split into two groups. The first one incorporates growth phenomena where thermal or annealed fluctuations affect the dynamics. The most famous example is the Kardar-Parizi-Zhang (KPZ) universality class characterized by the roughness exponent $\chi = 1/2$ and the dynamical exponent $z = 3/2$ in $1 + 1$ dimensions [5]. In the second group, an interface moves under the action of an external driving force F, which competes with pinning forces due to quenched medium inhomogeneities. At some critical force F_c the transition occurs from the totally pinned state to the state where the interface moves with a velocity v, which shows a power law decay $v \sim (F - F_c)^\theta$ when F approaches the critical point F_c from above [6].

The scaling properties of interfaces in the quenched case strongly depend on the medium isotropy. The isotropic medium produces a rough interface, which is believed to obey the quenched KPZ equation with a nonlinear term vanishing at the depinning threshold [7,8]. In the anisotropic medium, the quenched KPZ equation holds only for a definite orientation (hard direction) of the interface, yielding a divergent nonlinear term. The tilt from the hard direction generates the gradient term in the equation which breaks the translation invariance. The existence of such a term gives rise to another universality class, tilted interface (TI) class, characterized in $1 + 1$ dimensions by the exact exponents $\chi = 1/2$, $z = 1$, and $\theta = 1$ [7,8].

It has been noted that thermal fluctuations below the depinning threshold can initiate avalanches, which lead to infinitely slow creep of the interface [9]. Far below depinning threshold, the avalanches lead to local advances of finite interface segments, so that they can be considered under coarsening as a local annealed disorder in an appropriate time scale. Approaching the depinning threshold, the avalanches increase up to the system size and contribute to a global interface depinning which is controlled by the quenched disorder. It is the aim of this Letter to treat the effects of annealed and quenched disorders in a frame of a unified model.

Because of the well-known correspondence between growth processes in $(1 + 1)$ dimensions and one-dimensional lattice gases [1], we can interpret the avalanche dynamics of interfaces in terms of avalanches of particles similar to those appearing in one-dimensional random sandpile models [10]. Then, we can use the asymmetric avalanche process (ASAP) which has been recently proposed and solved by the Bethe ansatz method [11]. Having formal origin in the asymmetric exclusion process (ASEP) [12], the ASAP has completely different dynamical behavior specific for systems with avalanche dynamics which has been widely investigated in the context of self-organized criticality [13,14].

The ASAP was defined as follows [11]. Consider the system of p particles on a one-dimensional lattice of N sites with periodical boundary conditions, i.e., a ring of N sites. The system evolves by discrete-time steps according to the following rules: (a) If all particles occupy different sites of the lattice, then one randomly chosen particle moves one step to the right neighboring site with a fixed probability Δ for one time step; (b) if there are $n > 1$ particles at the same site, referred to as an active site, the diffusive motion is replaced by the avalanche dynamics: either all n particles move to the next right site at the next time step with probability μ_n or $n - 1$ particles move one step to the next right site with probability $1 - \mu_n$, while one particle remains unmoved. Thus, we use a random sequential update and separate in time the avalanche and single particle dynamics. The continuous time limit, $\Delta \to 0$, coincides with the case considered in [11], as only the terms of order Δ survive in the master equation. The dynamical rules (a) and (b) imply that at most one active site exists at any moment of time.

The interface problem appears if we associate the occupation number n_i on a site i with an interface height decrease $n_i = h(i) - h(i+1)$ so that the density of particles $\rho = p/N$ is the average tilt of the interface with respect to the original 1D lattice. The periodic boundary conditions with the tilted interface imply a helicoidal boundary condition for the interface, $h(i+N) = h(i) - p$. At every moment of time, the height of such an interface is a monotonously decreasing step function. The step of height $n = 1$ moves right diffusively (with probability Δ), whereas the steps of height $n > 1$ move like avalanches. A moving step, i.e., avalanche front, can increase by 1, merging with a step next to it, or decrease by 1, leaving a new step behind. The latter, happening with probability $1 - \mu_n$, mimics the action of random pinning forces.

In the infinite system, different choices of probabilities μ_n have been shown to lead to different regimes of particle flow, intermittent or continuous [11]. In the former case, finite avalanches are triggered by rare single particle jumps. As far as the avalanches are finite, we use the jump rate as a characteristic time scale, referred to as a diffusive timescale. To meet the continuous time picture of Ref. [11], we associate every discrete-time step with a continuous time interval $dt = \Delta/p$ with Δ approaching zero. Then the average velocity of the interface, $V \equiv \langle \partial_t h \rangle = \rho \langle s \rangle$, is given by the product of an average avalanche size, $\langle s \rangle$, i.e., average area attached to the interface during an avalanche, and the rate of avalanche initiation per site (equal to the average tilt) ρ. The large scale and long time properties of the ASAP, e.g., the variance of the interface height fluctuation, $w^2 \sim t/\sqrt{N}$, and its large deviation function, in the intermittent flow phase obtained from the Bethe ansatz solution are shown [15] to coincide with properties of the ASEP, which serves as a model for the discretized KPZ equation. Though these results are obtained from the Bethe ansatz solution for special choice of probabilities μ_n, which provides integrability, KPZ behavior is expected to hold for a much wider class of probabilities, provided that the avalanches remain finite.

The above quantities, however, become singular when the average tilt ρ reaches the point of transition to the continuous flow phase, ρ_c. For example, the average velocity, V, diverges with critical exponent α [11],

$$V \sim (\rho_c - \rho)^{-\alpha}. \tag{1}$$

The divergency of V means that the description based on the diffusive time scale fails in the thermodynamic limit. In this Letter, we obtain the exact V of finite interface and show that V grows together with the system size at the critical point and above. This means that avalanches lose their local character and cover the whole system. This situation is specific for the pinning-depinning phase transition above the critical point. To describe the interface in this case, one should look at the time evolution of a single avalanche at a time scale much faster than the diffusive one. The lattice model of such a kind belonging to the TI universality class has been first solved by Maslov and Zhang (MZ) at the critical point [10] by using the mapping of avalanche dynamics to the simple random walk problem. In this Letter we generalize this mapping to the more general Gillis random walk problem [16], which allows one to consider arbitrary values of α besides $\alpha = 2$ corresponding to the MZ model. Combining the exact formula for the velocity of the finite interface with random walk arguments we study the crossover between two types of critical behavior of interfaces and show which parameters are relevant near the transition point.

Consider first the stationary state of the system. The stationary probability, $P(C)$, of the particle configuration C on the lattice can be obtained from the balance equation

$$\sum_{\{C'\}} [T(C,C')P(C') - T(C',C)P(C)] = 0, \tag{2}$$

where $T(C,C')$ denotes the transition probability from the configuration C' to C and the summation is over all possible configurations of particles. For $T(C,C')$ specified by the dynamical rules described above the solution of Eq. (2) can be represented as a product of "one-site" factors [17]. As at most one active site exists in any configuration, its probability depends only on the number of particles in this site. Let $\mathcal{P}^{[n]}$ be the probability of any configuration with n particles at the active site. Then the solution of Eq. (2) is given by

$$\mathcal{P}^{[n+1]} = \mathcal{P}^{[n]} \mu_n / (1 - \mu_{n+1}), \tag{3}$$

where $n = 1, 2, \ldots, p-1$, and $\mu_1 = \Delta/p$. The only undetermined constant here is $\mathcal{P}^{[1]}$, the probability of stable configurations with no active site, which is defined from the normalization condition $\sum_{\{C\}} P(C) = 1$.

As it follows from Eq. (3), when n particles leave an active site at some step of an avalanche, all configurations of the remaining $p - n$ particles at the lattice occur equally likely. Therefore, the expected number $P_k(n)$ of events when n particles leave an active site at the kth step of an avalanche obey the following Markov equation:

$$P_{k+1}(n) = \sum_m P_k(m) w_{m,n} \tag{4}$$

with transition probabilities

$$w_{n-1,n} = \left(\rho - \frac{n-1}{N}\right)\mu_n,$$
$$w_{n,n} = \left(\rho - \frac{n}{N}\right)(1 - \mu_{n+1}) + \left[1 - \left(\rho - \frac{n}{N}\right)\right]\mu_n,$$
$$w_{n+1,n} = \left[1 - \left(\rho - \frac{n+1}{N}\right)\right](1 - \mu_{n+1})$$

for $n > 1$, $w_{1,2} = (\rho - \frac{1}{N})\mu_2$, $w_{1,1} = (\rho - \frac{1}{N})(1 - \mu_2)$, $w_{2,1} = [1 - (\rho - \frac{2}{N})](1 - \mu_2)$, and $w_{m,n} = 0$ in all the other cases. The transition probabilities $w_{m,n}$ correspond to the events when m particles flow into a site and n particles flow out, generalizing those in [11] to the case of the finite lattice. Proceeding parallel to [11], we obtain the total expected number of jumps of n particles during the whole avalanche $P(n) = \sum_{k=1}^{\infty} P_k(n)$,

$$P(n) = \frac{N\Gamma(p)\Gamma(N - p + 1)}{\Gamma(p - n + 1)\Gamma(N - p + n + 1)} \prod_{j=2}^{n} \frac{\mu_j}{1 - \mu_j}, \quad (5)$$

for $n > 1$. The expected total number of particles spilled during an avalanche, i.e., the average avalanche size, is given by

$$\langle s \rangle = V/\rho = \sum_{n=1}^{p} nP(n). \quad (6)$$

Consider the asymptotic limit $N \to \infty$, $p \to \infty$, $p/N = \rho = \text{const}$. To this end, we have to specify the toppling probabilities μ_n for large n. We suppose that μ_n tends to a constant μ_∞ when $n \to \infty$ having the following $(1/n)$ expansion: $\mu_n/(1 - \mu_n) = \mu_\infty/(1 - \mu_\infty)[1 + (\alpha - 2)/n + O(1/n^2)]$. Then, for all positive α, the sum in Eq. (6) is finite if the density is less than the critical value $\rho_c = 1 - \mu_\infty$. The parameter α is introduced in such a way that the expansion of Eqs. (5) and (6) in n/N results in the power law of Eq. (1). The $1/N$ correction to this law is of interest, as it determines the behavior of the nonlinear coefficient λ in the KPZ equation [2], which depends on the tilt diverging at the critical point

$$\lambda \sim N[V(\infty) - V(N)] \sim (\rho_c - \rho)^{-\alpha - 2}. \quad (7)$$

The behavior of $V(\infty)$ is also related to λ through a nonuniversal constant a, $V(\infty) \sim \lambda/a$, which plays the role of a small length cutoff in the KPZ equation. In our case, we have $a \sim (\rho_c - \rho)^{-2}$. Considering the expansion of V in powers a/N, we see that the requirement of the parameter $r = a/N$ to be small gives a limit of applicability of the undercritical expansion and, hence, of KPZ description. In such a way, r is expected to be the control parameter of the transition, which is confirmed below in a different context.

To probe into the whole phase space, we approximate Γ functions in Eq. (5) by the Stirling formula and replace the sum in Eq. (6) by the integral. The equation for the saddle point, $[\ln P(n_0)]' = 0$, results in

$$n_0 \simeq N(\rho - \rho_c), \qquad [\ln P(n_0)]'' \simeq [N\rho(1 - \rho)]^{-1}. \quad (8)$$

Hence, when $\rho < \rho_c$, only the region $n \ll N$ contributes the sum in Eq. (6), which confirms the validity of the n/N expansion used. The location of the saddle point at n_0 has due to Eq. (8) a transparent physical meaning. It is the height of an avalanche front spreading through the lattice, which compensates the excess slope of the interface maintaining it at the critical tilt ρ_c [8]. The evaluation of the integral in the saddle point approximation for $\rho > \rho_c$ gives the leading part of V, exponentially growing with N

$$V \sim (\rho - \rho_c)^{\alpha - 1} N^{\alpha - 1/2} \left[\left(\frac{\rho}{\rho_c} \right)^{\rho} \left(\frac{1 - \rho}{1 - \rho_c} \right)^{1 - \rho} \right]^N. \quad (9)$$

At the critical tilt, ρ_c, the average avalanche size reveals the power law dependence on the system size

$$V_c \sim N^{\alpha/2}. \quad (10)$$

The crossover from subcritical to supercritical regime through the critical point can be viewed in the function obtained from another expansion of Eq. (5) for $1 \ll n_0 \ll N$, namely, $V \sim V_c g_\alpha(u)$, where

$$g_\alpha(u) = \frac{2e^{u^2}}{\Gamma(\alpha/2)} \int_{-u}^{\infty} dx (x + u)^{\alpha - 1} e^{-x^2} \quad (11)$$

is a function depending on the parameter α and variable $u = \sqrt{N(1 - \rho)\rho/2} \ln \frac{(1 - \rho_c)\rho}{(1 - \rho)\rho_c}$. Near the critical point $u \sim (\rho - \rho_c)\sqrt{N}$, or in terms of the parameter r discussed above, $u \sim -1/\sqrt{r}$. The function $g_\alpha(u)$ is equal to 1 when $u = 0$ and decays as $|u|^{-\alpha}$ when $u \to -\infty$, thus eliminating the dependence of V on N in the subcritical regime. It is clear from the results obtained that the limits $\rho \to \rho_c$ and $N \to \infty$ do not commute. Then, the value of u serves as a characteristic distance from the critical point, i.e., a parameter indicating either the system is in the subcritical (KPZ) or critical (TI) regime.

As we have noted, at the critical point and above the dynamics of avalanche in the fast time scale should be considered to describe properties of the interface. According to [8], the TI universality class is characterized by fronts moving parallel to the average tilt, being characterized by the KPZ equation in $(d - 1)$ transverse directions. In the 1D case, the $0 + 1$ dimensional KPZ equation is reduced to a noise term only, which corresponds to simple random walks of height of the active site. Let us consider the form of transition probabilities in Eq. (4) for $\rho = \rho_c$. One can see that the height of an active site in the ASAP also performs the simple random walk when $\alpha = 2$. However, for $\alpha \neq 2$ the normalized transition probabilities, defined as $p(n \to n \pm 1) = w_{n,n\pm 1}/(w_{n,n+1} + w_{n,n-1})$, contain the nonuniform bias term, $(\alpha - 2)/2n$, decreasing with the distance n from the origin. This term, first considered by Gillis [16], is the only relevant term of the expansion, $p(n \to n \pm 1) = 1/2[1 \pm (\alpha - 2)/(2n) + O(1/n^2)]$, which changes the asymptotical behavior of first return time of the random walk. Depending on whether α is larger or less than two, it can be either positive or negative, respectively, enhancing or suppressing the avalanche spreading.

As usual, we assume the following scaling ansatz for avalanche size and time distribution $P(s) = s^{-\tau} g(s/s_0)$ and $P(t) = t^{-\tau_t} g(t/t_0)$ in the vicinity of the critical point, where $g(x)$ is a scaling function, and s_0, t_0 are time and size cutoffs. The time cutoff t_0 plays also the role of the avalanche correlation length. Knowing asymptotics for mean conditional time of the first return for the Gillis random walk [18], we can obtain the avalanche time critical exponent, $\tau_t = 5/2 - \alpha/2$ for $1 \leq \alpha < 3$. Considering the avalanche size statistics, we should note that unlike the exponent τ_t, the dimension of avalanches D does not depend on α and coincides with the unbiased case, which can be directly checked by multiplying the original equation, Eq. (4), by n^2 and summing by parts. This yields $\langle n^2 \rangle \sim t$ and hence $D = 3/2$ and $\tau = 2 - \alpha/3$. The other critical exponents characterizing underlying interface dynamics remain unchanged comparing to the unbiased MZ case. Particularly, the exponent characterizing the correlation length below the critical point, defined in our case by $t_0 \sim (\rho_c - \rho)^{-\nu}$, can be uniquely fixed as $\nu = 2$ from the relation between the average avalanche size below ρ_c, Eq. (1), with the correlation length, $V \sim t_0^{D(2-\tau)}$. Hence, the upper bound of the avalanche length t_0 coincides with the small length cutoff a of the KPZ equation, i.e., below the critical point it is that parameter, which defines the space scale of continuous KPZ description. The relation (10) shows that t_0 approaches the system size, $t_0 \sim N$, at the critical point irrespectively of α, which confirms the dynamical exponent $z = 1$. Thus, the control parameter u in Eq. (11) can be treated as a ratio of the avalanche correlation length to its critical value. The exponent $\theta = 1$ just above the depinning threshold follows from the value of n_0, Eq. (8), which gives a characteristic increase of the interface height after one avalanche passage proportional to $(\rho - \rho_c)$. The roughness exponent is $\chi = 1/2$ due to equiprobability of stationary particle configurations, $\mathcal{P}^{[1]}(C) = \text{const}$, established above [19], which is equivalent to the Gaussian measure for the gradient of the interfacial height in the continuum limit. At the critical point, it can also be obtained from the dimension of the random walk $\chi = D - d$ [8,10].

We should note that the scaling relation $\tau = 1 + (d - 1/\nu)/D$ between exponents ν and τ [10] does not hold in our case for $\alpha \neq 2$. This relation reflects the fact that the interface below the depinning threshold is frozen in the metastable state where the width and avalanche size are expressed through the same correlation length [4]. In our case, however, the interface below the critical point comes to the stationary state due to annealed disorder, irrelevant of the details of the avalanche dynamics, with the interfacial width bounded only by the system size. Thus, the symmetry responsible for the scaling relation is clearly absent in our case.

The above results are valid for $1 \leq \alpha < 3$. For $\alpha < 1$, the average avalanche size is finite, and universal behavior is lost. For $\alpha > 3$, the Gillis random walk becomes transient; i.e., its return to the origin is not definite. When $\alpha = 3$, logarithmic corrections to power laws should be taken into account. These cases will be studied elsewhere.

In summary, we have found that the asymmetric avalanche process (ASAP) [11] incorporates the annealed KPZ and quenched TI interface dynamics for $\rho < \rho_c$ and $\rho > \rho_c$, respectively. All critical exponents characterizing the interface behavior in both classes are obtained exactly and shown to coincide with those known before. Nevertheless, the exact calculation of the average avalanche size and random walk arguments show that the critical exponents of avalanche distributions are continuous functions of the parameter α responsible for the asymptotics of toppling probabilities.

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*Corresponding author.
Electronic address: huck@phys.sinica.edu.tw

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