Planar Amplitudes in Maximally Supersymmetric Yang-Mills Theory

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The collinear factorization properties of two-loop scattering amplitudes in dimensionally regulated $N = 4$ super-Yang-Mills theory suggest that, in the planar ('t Hooft) limit, higher-loop contributions can be expressed entirely in terms of one-loop amplitudes. We demonstrate this relation explicitly for the two-loop four-point amplitude and, based on the collinear limits, conjecture an analogous relation for n-point amplitudes. The simplicity of the relation is consistent with intuition based on the anti–de Sitter/conformal field theory correspondence that the form of the large-N_c L-loop amplitudes should be simple enough to allow a resummation to all orders.

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Four-dimensional quantum field theories are extremely intricate, and generically have complicated perturbative expansions in addition to nonperturbative contributions to physical quantities. Gauge theories are interesting in that numerous cancellations occur. This renders perturbative computations more tractable, and their results simpler, than one might otherwise expect. The Maldacena conjecture [1] implies that a special gauge theory is simpler yet: the 't Hooft (planar) limit of maximally supersymmetric four-dimensional gauge theory, or $N = 4$ super-Yang-Mills (MSYM) theory. The conjecture states that the strong coupling limit of this conformal field theory (CFT) is dual to weakly coupled gravity in five-dimensional anti–de Sitter (AdS) space. The AdS/CFT correspondence is remarkable in taking a seemingly intractable strong coupling problem in gauge theory and relating it to a weakly coupled gravity theory, which can be evaluated perturbatively. There have been multiple quantitative tests of this correspondence, using observables protected by supersymmetry (see, e.g., Ref. [2]). Because of the different domains of validity of coupling expansions on the gauge and gravity sides, quantitative comparisons involving unprotected quantities rely at present on an additional expansion parameter, such as in the large-J ("spin") limit of Berenstein-Maldacena-Nastase operators [3,4].

In this latter context, the AdS/CFT correspondence can be used to motivate a search for patterns in the perturbative expansion of planar MSYM theory. Intuitively, observables in the strongly coupled limit of this theory should be relatively simple because of the weakly coupled gravity interpretation. Yet infinite orders in the perturbative expansion, as well as nonperturbative effects, contribute to the strong coupling limit. How might such a complicated expansion organize itself into a simple result? For quantities protected by supersymmetry, nonrenormalization theorems, or zeros in the perturbative series, are one possibility. Another possibility, for unprotected quantities, is some iterative perturbative structure allowing for a resummation. There have been some hints of an iterative structure developing in the correlation functions of gauge-invariant composite operators [5], but the exact structure, if it exists, is not yet clear.

Amplitudes for scattering of on-shell (massless) quanta—gluons, gluinos, etc.—are examples of particular interest because of their importance in QCD applications to collider physics. Although the Maldacena conjecture does not directly refer to on-shell amplitudes, we expect the basic intuition, that the perturbation expansion should have a simple structure, to hold nonetheless. Indeed, the simplicity of one- and two-loop amplitudes in MSYM has allowed their computation to predate corresponding QCD calculations [6,7].

Perturbative amplitudes in four-dimensional massless gauge theories are not finite, but contain infrared singularities due to soft and collinear virtual momenta. The divergences can be regulated using dimensional regularization with $D = 4 - 2\epsilon$. The resulting poles in ϵ begin at order $1/\epsilon^{2L}$ for L loops, and are described by universal formulas valid for MSYM, QCD, etc. [8]. To preserve supersymmetry we use the four-dimensional helicity scheme [9] variant of dimensional regularization, which is a close relative of dimensional reduction [10]. The infrared divergences turn out to have precisely the iterative structure we shall find in the full $N = 4$ amplitudes; thus, they provide useful guidance toward exhibiting such a structure.

Infrared divergences generically prevent the definition of a textbook S matrix in a nontrivial conformal field theory such as MSYM. For the dimensionally regulated S

matrix elements we discuss, the regulator explicitly breaks the conformal invariance. However, once the universal infrared singularities are subtracted, the four-dimensional limit of the remaining terms in the amplitudes may be taken, allowing an examination of possible connections to the Maldacena conjecture.

These finite remainders are relevant for computing "infrared-safe" observables in QCD, in which the divergent parts of virtual corrections cancel against real-radiative contributions (not discussed here) to produce finite perturbative results [11]. The finite remainders should also be related to perturbative scattering matrix elements for appropriate coherent states (see, e.g., Ref. [12]). The connection to the S matrix for the true asymptotic states of the theory, such as the hadrons of QCD, is, of course, nontrivial.

In MSYM, there are other hints that higher-loop amplitudes are related in a simple way to the one-loop ones. In particular, the integrands of the amplitudes (prior to evaluation of loop-momentum integrals) have a simple iterative structure [7]. Furthermore, the one-loop amplitudes have a relatively simple analytic structure, which has allowed their determination to an arbitrary number of external legs for configurations with maximal helicity violation [13] and up to six external legs for all helicities [14]. Unitarity then suggests that higher-loop amplitudes may also have a relatively simple analytic structure.

In this Letter we present direct evidence that this intuition is correct for the planar amplitudes of MSYM. A number of powerful techniques are available to compute them. These include the unitarity-based method [7,13,14], recently developed multiloop integration methods (see Ref. [15] and references therein), and the imposition of constraints from required behavior as the momenta of two external legs become collinear [16]. Here we shall express the explicit form for the four-point $N = 4$ amplitude at two loops, in terms of the one-loop amplitude, using previous results [7,17]. In addition, we present the two-loop splitting amplitude in planar MSYM, computed elsewhere, which summarizes the behavior of amplitudes as the momenta of two legs become collinear. We use the latter to provide evidence that the relationship between the two-loop and one-loop amplitudes continues to hold for an arbitrary number of external legs.

The leading-N_c contributions to the L-loop SU(N_c) gauge-theory n-point amplitudes may be written as

$$\mathcal{A}_n^{(L)} = g^{n-2}\left[\frac{2e^{-\epsilon\gamma}g^2N_c}{(4\pi)^{2-\epsilon}}\right]^L \sum_\rho \mathrm{Tr}(T^{a_{\rho(1)}}T^{a_{\rho(2)}}\cdots T^{a_{\rho(n)}}) \times A_n^{(L)}(\rho(1),\rho(2),\ldots,\rho(n)), \tag{1}$$

where the sum is over noncyclic permutations of the external legs, and we have suppressed the momenta and helicities k_i and λ_i, leaving only the index i as a label.

The color-ordered amplitudes $A_n^{(L)}(1,2,\ldots,n)$ satisfy simple properties as the momenta of two color-adjacent legs k_a, k_b become collinear,

$$A_n^{(L)}(\ldots,a^{\lambda_a},b^{\lambda_b},\ldots) \to \sum_{l=0}^{L}\sum_{\lambda=\pm}\mathrm{Split}_{-\lambda}^{(l)}(z;a^{\lambda_a},b^{\lambda_b}) \times A_{n-1}^{(L-l)}(\ldots,P^\lambda,\ldots). \tag{2}$$

The index l sums over the different loop orders of contributing splitting amplitudes $\mathrm{Split}_\lambda^{(l)}$, while λ sums over the helicities of the fused leg $k_P = -(k_a + k_b)$, where z is the momentum fraction of k_a, $k_a = -zk_P$. The two-loop version of this formula is sketched in Fig. 1. The splitting amplitudes are universal and gauge invariant. Formula (2) provides a strong constraint on amplitudes; for example, it has been used to fix the form of a number of one-loop n-point amplitudes [13,14,16].

At tree level, the splitting amplitudes $\mathrm{Split}_\lambda^{(0)}$ are the same in MSYM as in QCD. Supersymmetry Ward identities [18] relate different helicity amplitudes in MSYM, implying that the loop splitting amplitudes may be expressed in terms of the tree-level ones [13],

$$\mathrm{Split}_{-\lambda_P}^{(L)}(1^{\lambda_1},2^{\lambda_2}) = r_S^{(L)}\mathrm{Split}_{-\lambda_P}^{(0)}(1^{\lambda_1},2^{\lambda_2}), \tag{3}$$

where $r_S \equiv r_S^{(L)}(\epsilon;z,s=(k_1+k_2)^2)$ is independent of λ, λ_1, λ_2. Similarly defining the scattering amplitude ratios $M_n^{(L)}(\epsilon) \equiv A_n^{(L)}/A_n^{(0)}$, we obtain in collinear limits

$$M_n^{(1)}(\epsilon) \to M_{n-1}^{(1)}(\epsilon) + r_S^{(1)}(\epsilon), \tag{4}$$

$$M_n^{(2)}(\epsilon) \to M_{n-1}^{(2)}(\epsilon) + r_S^{(1)}(\epsilon)M_{n-1}^{(1)}(\epsilon) + r_S^{(2)}(\epsilon). \tag{5}$$

The $N = 4$ one-loop splitting amplitudes have been calculated to all orders in ϵ [19], with the result

$$r_S^{(1)}(\epsilon;z,s) = \frac{\hat{c}_\Gamma}{\epsilon^2}\left(\frac{\mu^2}{-s}\right)^\epsilon\left[-\frac{\pi\epsilon}{\sin(\pi\epsilon)}\left(\frac{1-z}{z}\right)^\epsilon + 2\sum_{k=0}^{\infty}\epsilon^{2k+1}\mathrm{Li}_{2k+1}\left(\frac{-z}{1-z}\right)\right], \tag{6}$$

where Li_n is the nth polylogarithm,

$$\hat{c}_\Gamma = \frac{e^{\epsilon\gamma}}{2}\frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}, \tag{7}$$

and γ is Euler's constant.

We have calculated the two-loop, leading-N_c, $N = 4$ splitting amplitudes through $\mathcal{O}(\epsilon^0)$ using the method of Ref. [20] with the result

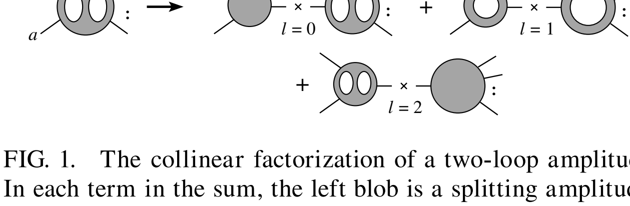

FIG. 1. The collinear factorization of a two-loop amplitude. In each term in the sum, the left blob is a splitting amplitude, and the right blob an $(n-1)$-point scattering amplitude.

$$r_S^{(2)}(\epsilon; z, s) = \tfrac{1}{2}(r_S^{(1)}(\epsilon; z, s))^2 + f(\epsilon) r_S^{(1)}(2\epsilon; z, s), \qquad (8)$$

with $f(\epsilon) \equiv [\psi(1-\epsilon) - \psi(1)]/\epsilon = -(\zeta_2 + \zeta_3\epsilon + \zeta_4\epsilon^2 + \cdots)$, $\psi(x) \equiv (d/dx)\ln\Gamma(x)$, $\psi(1) = -\gamma$.

The infrared singularities of leading-N_c MSYM at one and two loops can be extracted from more general studies, notably Ref. [8]. At one loop, the divergences are given by

$$C_n^{(1)}(\epsilon) = -\frac{e^{\epsilon\gamma}}{2\Gamma(1-\epsilon)}\frac{1}{\epsilon^2}\sum_{i=1}^{n}\left(\frac{\mu^2}{-2k_i \cdot k_{i+1}}\right)^{\epsilon}. \qquad (9)$$

The two-loop divergences, in the four-dimensional helicity scheme, are [8,21]

$$C_n^{(2)}(\epsilon) = \frac{1}{2}(C_n^{(1)}(\epsilon))^2 + C_n^{(1)}(\epsilon)F_n^{(1)}(\epsilon) - (\zeta_2 + \epsilon\zeta_3)\frac{e^{-\epsilon\gamma}\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)}C_n^{(1)}(2\epsilon). \qquad (10)$$

The finite remainder is defined by subtraction,

$$F_n^{(L)}(\epsilon) = M_n^{(L)}(\epsilon) - C_n^{(L)}(\epsilon). \qquad (11)$$

Note that $C_n^{(L)}(\epsilon)$ contains some finite terms as well.

We now present evidence that through $\mathcal{O}(\epsilon^0)$ the two-loop planar amplitudes are related to one-loop ones via

$$M_n^{(2)}(\epsilon) = \frac{1}{2}(M_n^{(1)}(\epsilon))^2 + f(\epsilon)M_n^{(1)}(2\epsilon) - \frac{5}{4}\zeta_4. \qquad (12)$$

Note the similarity of our ansatz to the two-loop splitting amplitude (8), as well as to the infrared subtraction (10).

The one-loop four-point amplitude in MSYM [6] is given in terms of the scalar box integral depicted in Fig. 2(a). Expanding the result in ϵ yields

$$\begin{aligned} M_4^{(1)}(\epsilon) = \hat{c}_\Gamma\bigg\{ &-\frac{2}{\epsilon^2}\left(\frac{\mu^2}{-s}\right)^{\epsilon} - \frac{2}{\epsilon^2}\left(\frac{\mu^2}{-t}\right)^{\epsilon} + \left(\frac{\mu^2}{u}\right)^{\epsilon}\bigg[\frac{1}{2}[(X-Y)^2 + \pi^2] + 2\epsilon\left(\mathrm{Li}_3(x) - X\mathrm{Li}_2(x) - \frac{X^3}{3} - \frac{\pi^2}{2}X\right) \\ &- 2\epsilon^2\bigg(\mathrm{Li}_4(x) + Y\mathrm{Li}_3(x) - \frac{X^2}{2}\mathrm{Li}_2(x) - \frac{X^4}{8} - \frac{X^3 Y}{6} + \frac{X^2Y^2}{4} \\ &- \frac{\pi^2}{4}X^2 - \frac{\pi^2}{3}XY - 2\zeta_4\bigg) + (s \leftrightarrow t)\bigg] + \mathcal{O}(\epsilon^3)\bigg\}, \end{aligned} \qquad (13)$$

where $s = (k_1 + k_2)^2$, $t = (k_1 + k_4)^2$, $u = -s - t$, $x = -s/u$, $y = -t/u$, $X = \ln x$, and $Y = \ln y$. For the four-point case, the $\epsilon \to 0$ limit of the finite remainder (11) is

$$F_4^{(1)}(0) = \frac{1}{2}\ln^2\left(\frac{-s}{-t}\right) + \frac{\pi^2}{2}. \qquad (14)$$

In Ref. [7] the two-loop $N = 4$ amplitude was presented in terms of a double-box scalar integral depicted in Fig. 2(b), plus its image under the permutation $s \leftrightarrow t$. Reference [17] provides the explicit value of this integral, through $\mathcal{O}(\epsilon^0)$, in terms of polylogarithms. Inserting this value, we obtain precisely the result (12) with $n = 4$. The equality requires the use of polylogarithmic identities and nontrivial cancellation of terms between the two integrals. Terms through $\mathcal{O}(\epsilon^2)$ in $M_4^{(1)}$ contribute at $\mathcal{O}(\epsilon^0)$ in $M_4^{(2)}$, since they can multiply $1/\epsilon^2$ terms.

Subtracting the two-loop infrared divergence given in Eq. (10) from our calculated expression yields

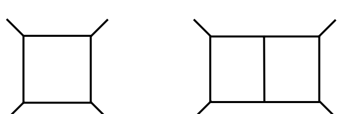

FIG. 2. The scalar integral functions appearing in the (a) one- and (b) two-loop four-point $N = 4$ amplitudes.

$$F_4^{(2)}(0) = \frac{1}{2}[F_4^{(1)}(0)]^2 - \zeta_2 F_4^{(1)}(0) - \frac{21}{8}\zeta_4, \qquad (15)$$

expressed in terms of the one-loop finite remainder (14).

For $n \geq 5$ legs, we examine the properties as external momenta become collinear, using Eq. (2). Applying the one-loop collinear behavior (4) to the ansatz (12), we have

$$M_n^{(2)}(\epsilon) \to \frac{1}{2}[M_{n-1}^{(1)}(\epsilon) + r_S^{(1)}(\epsilon)]^2 + f(\epsilon)[M_{n-1}^{(1)}(2\epsilon) + r_S^{(1)}(2\epsilon)] - \frac{5}{4}\zeta_4, \qquad (16)$$

which is consistent with the required two-loop collinear properties (5), using Eq. (8). Although severely constrained, amplitudes are not uniquely defined by their collinear limits [13]. Thus, Eq. (12) remains unproven for $n \geq 5$. The direct computation of the two-loop five-point function seems feasible, and would provide an important test of the ansatz.

We investigated two potential extensions of the relation (12), each with negative results.

(1) We examined the nonplanar extension by computing the subleading-color two-loop finite remainders, analogous to $F_4^{(2)}(0)$. These terms contain polylogarithms, and hence cannot be written in terms of one-loop finite remainders, unlike the planar Eq. (15). Thus, the nonplanar terms do not appear to have a structure analogous to

Eq. (12), in line with heuristic expectations from the Maldacena conjecture.

(2) For the four-point amplitude, we find that Eq. (12) is *not* satisfied at $\mathcal{O}(\epsilon)$, due to polylogarithmic obstructions. Hence, the relation holds only as $D \to 4$, i.e., where the theory becomes conformal.

The possibility of resumming perturbative expansions in MSYM may also have relevance for QCD. QCD may be viewed as containing a "conformal limit" (e.g., MSYM) plus conformal-breaking terms. This perspective has had practical impact on topics ranging from the Crewther relation to exclusive processes [22]. We note that $N = 4$ amplitudes can be obtained directly from QCD amplitudes by adjusting the number and color of states circulating in the loop: starting from the two-loop QCD amplitudes of Ref. [21] and substituting for the "spin index dimension" $D_s = 4 - 2\epsilon\delta_R \to 10$ and for the color Casimirs $C_F \to C_A$, $T_R N_f \to 2C_A$, one obtains the two-loop MSYM amplitudes. (These modifications effectively give $D = 10$, $N = 1$ super-Yang-Mills theory, truncated to $D = 4$, which is $N = 4$ super-Yang-Mills theory. See Eq. (6.5) of Ref. [21] for an analogous conversion to $D = 4$, $N = 1$ super-Yang-Mills amplitudes.)

A number of open questions deserve further study. At two loops, the $N = 4$ planar ansatz should be checked for at least the five-point case. At higher-loop order, the intuition described in the introduction suggests a continuation of the iterative structure found at two loops, possibly enabling a resummation of perturbative contributions. Thus, we expect that higher-loop planar $N = 4$ amplitudes will be "polynomial" functions of the one-loop amplitudes. Indeed, the known form of three-loop infrared divergences [8] provides some confirmation of this. Recent advances should make possible explicit evaluation of the leading-color three-loop four-point amplitude, using known expressions for the integrand [7]. [One of the two three-loop integrals needed has already been computed through $\mathcal{O}(\epsilon^0)$ in terms of generalized polylogarithms [23].] One would also like to identify a symmetry (presumably related to superconformal invariance) and associated Ward identity responsible for restricting amplitudes to be iterations of the one-loop amplitude; recall that the relation between the two-loop and one-loop amplitudes holds only near $D = 4$ where the theory is conformal. We are optimistic that an understanding of the amplitudes of $N = 4$ super-Yang-Mills theory will lead to new insight into consequences of the AdS/CFT correspondence.

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