Scaling of the Turbulence Transition Threshold in a Pipe

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We report the results of an experimental investigation of the transition to turbulence in a pipe over approximately an order of magnitude range in the Reynolds number Re. A novel scaling law is uncovered using a systematic experimental procedure which permits contact to be made with modern theoretical thinking. The principal result we uncover is a scaling law which indicates that the amplitude of perturbation required to cause transition scales as $O(\mathrm{Re}^{-1})$.

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The puzzle of why the flow of a fluid along a pipe is typically observed to change from laminar to turbulent as the flow rate is increased is a well-known paradox of hydrodynamic stability theory. The issue is both of deep scientific and engineering interest since most pipe flows are turbulent in practice even at modest flow rates. All theoretical and numerical work indicates that the flow is linearly stable [1]. It is natural to assume that finite amplitude perturbations are therefore responsible for triggering turbulence and these become more important as the nondimensionalized flow rate, the Reynolds number, Re, increases [2]. (Here $\mathrm{Re} = Ud/\nu$, where U is the mean speed of the flow, d is the diameter of the pipe, and ν is the kinematic viscosity of the fluid.)

A question which may be asked is, if $\epsilon = \epsilon(\mathrm{Re})$ denotes the minimal amplitude of all finite perturbations that can trigger transition, and if ϵ scales with Re according to

$$\epsilon = O(\mathrm{Re}^{\gamma}) \tag{1}$$

as $\mathrm{Re} \to \infty$, then what is the exponent γ [3]? A negative value of γ is anticipated and one substantially less than zero would indicate that the sensitivity of the laminar flow increases rapidly with Re, i.e., the basin of attraction of the laminar fixed point diminishes rapidly as Re increases. Current estimates for γ suggest that for shear flows it lies within the range $-1 \geq \gamma \geq -7/4$ from various model studies [4,5] and numerical simulations [6]. An exponent strictly less than -1 would indicate the importance of transient growth while -1 is expected from a simple balance between nonlinear advection and viscous dissipation [7]. A significant challenge is to relate this theoretical concept to observation in a quantitative manner although some limited data are available [8]. In this Letter we provide evidence from a novel experiment which suggests a way forward and provides striking evidence for an exponent of -1 which points to a generic transition [7].

Reynolds [9] showed that when Re was greater than approximately 2000 turbulent flow was the typical flow state. Importantly, he also showed that minimizing inlet disturbances enabled laminar flow to be maintained to higher flow rates. This finding has been extended in modern times to Reynolds numbers of $\approx$100 000 [10] in transient flows by taking extraordinary care.

The process whereby turbulence arises is not understood either in outline or in detail, and any advance towards an understanding of the fundamentals involved will have widespread impact on flows of practical interest. For example, the flows in oil and gas pipelines are often run inefficiently turbulent to avoid the large pressure fluctuations of the transitional regime. Moreover, the control of turbulence is a dream of many practitioners, just as an understanding of turbulence is the desire of many scientists.

In general terms, pipe flow may be considered as a nonlinear dynamical system $d\mathbf{u}/dt = f(\mathbf{u}, \mathrm{Re})$ which represents the Navier-Stokes equations subject to appropriate forcing and boundary conditions. There is one linearly stable fixed point for all Re and another state or "attractor," turbulence, when $\mathrm{Re} > \mathrm{Re}_c$. Hence when $\mathrm{Re} < \mathrm{Re}_c$ all initial conditions are attracted to the laminar state which is the global attractor for the system. When $\mathrm{Re} \gg \mathrm{Re}_c$ nearly all initial conditions give rise to turbulence and the laminar state becomes a local attractor. In practice, $\mathrm{Re}_c \approx 2000$ so that all disturbances will decay as $t \to \infty$ for values of Re smaller than this. These arguments are consistent both with Reynolds's original observations and modern experimental results [11,8,12,13]. Almost all experimental studies of the problem have been concerned with pressure gradient driven flows so that large fluctuations in flow rate and hence Re can, in principle, occur upon transition. In one exception to this [13] a constant mass flux system is used where the flow is pulled by a piston and this accurately fixes Re. Impulsive perturbations are used to produce a finite amplitude stability curve such that disturbances with amplitudes greater than a threshold produce turbulence while smaller ones decay downstream. The results are consistent with those obtained with pressure driven systems [11,8,12] so that localized "puffs" and "slugs" are found at low Re and fully developed turbulence at larger flow rates.

Modern theoretical research may be broadly split into two approaches. In one, initially small disturbances on the laminar state grow in a transient phase [3,14] until they reach a sufficiently large amplitude that nonlinear effects become important. These ideas have been explored for low-dimensional models [15] and applied to plane Poiseuille flow [16], and scaling laws for the amplitude of the perturbation as a function of Re have been provided. An alternative point of view [17] is that the turbulent state originates from instabilities of a finite amplitude solution which is disconnected from the base state. The basin of attraction of the turbulent state grows with Re so that any small finite perturbation will kick the laminar solution towards it. Such solutions of the Navier-Stokes equations are known to exist in other flows [18–20], but their existence has not yet been shown in pipe flows.

In drawing a connection between experimental observations and predictions from models the notion of a perturbation needs to be clarified. In models, the temporal and the spatial form of the perturbation are precise. On the other hand, experimentalists inject and/or subtract fluid through slits or holes in an attempt to mimic the mathematical process. The perturbation can be either periodic [8,12] or impulsive [13], but specifying key measures such as the scale of the amplitude is difficult. Indeed, resolving the pertinent part of the physical perturbation which gives rise to transition is also difficult [12]. We have devised a novel type of perturbation which permits a scaling analysis and thereby allows a closer connection to be made with theory.

The experimental system can be regarded as a large syringe which pulls water at a fixed mass flux along a precision bore tube. We have two such experimental facilities in our laboratory, and we only outline details of the new rig as the first system has been described previously [13]. The pipe consisted of a 20 mm diameter Perspex tube which was 15.7 m in length and constructed from 105 machined sections each of which was 150 mm long. The sections were fitted together and aligned with a laser on a steel base. Still water was drawn from a tank through a trumpet shaped inlet into the tube and a similar expansion connected the 260 mm diameter piston to the tube. The flow state was monitored using flow visualization and recorded at various spatial locations using video cameras whose images were stored for further processing. Laminar flow was achieved for $\mathrm{Re} \leq 24\,000$ verifying the quality of the construction. We report results only for $\mathrm{Re} \leq 18\,000$, and the flow was fully developed at the disturbance injection point up to $\mathrm{Re} = 16\,000$. A schematic diagram showing the experimental arrangement for the "long pipe" is given in Fig. 1, and the shorter pipe was constructed in a very similar way but was 190 diameters long.

The stability of the flow was probed using a perturbation which was applied to a sufficient number of diameters from the inlet to ensure fully developed flow over the Re ranges investigated. This distance was 75 and 530 pipe diameters for the short and long pipes, respectively. A single boxcar pulse of fluid was injected tangentially into the flow via a ring of six equally spaced 0.5 mm holes. The injection system contained two high speed solenoid valves with switching times of ≈ 1 ms, and the rise and fall times of the perturbation were limited by the inertia of the piston. The quantities of fluid injected were in the range 0.01% to 0.1% of the total volume flux where the larger values were required to cause transition at smaller Re. This novel injection system enabled us to vary the duration and amplitude of the perturbation independently. We show in Fig. 2 a typical pressure trace of a perturbation of magnitude Δp and width Δt. It can be seen that a reasonable approximation to a boxcar function is achieved with relatively small amplitude ringing at the switching times. The pressure was measured at a single location to illustrate the form of the perturbation and measurement of pressure gradient in this format remains a technical challenge. The displaced volume flux Φ_{inj} from the injector is used in our definition of the amplitude of the perturbation. The duration of the injection sets the spatial extent of the disturbed flow, and the relative volume flux enables a direct connection with theory. In principle, the perturbation will have a global effect on the flow field, but checks using injection and suction [13] show that it is localized in practice.

Injecting the disturbance in this way permits the amplitude and duration of the perturbation to be varied

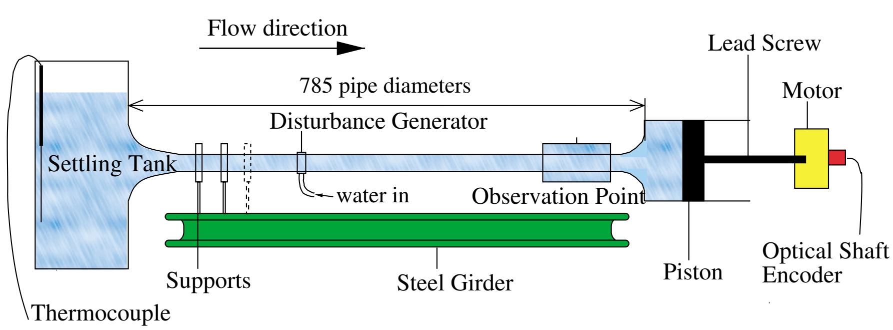

FIG. 1 (color online). Schematic of the long pipe experimental system.

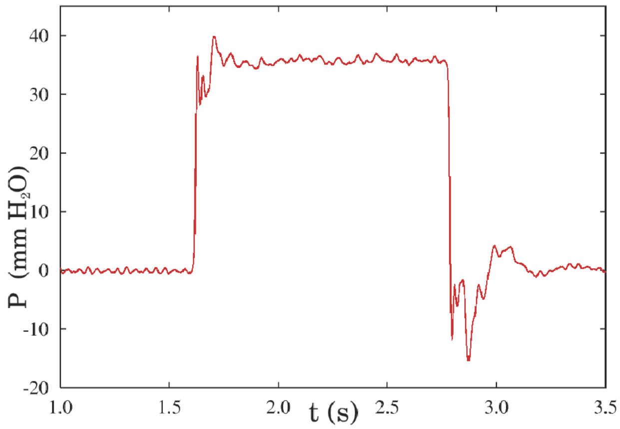

FIG. 2 (color online). A typical pressure time series for a perturbation of amplitude $\Delta p = 37 \pm 1$ mm H_2O and width $\Delta t = 1.2 \pm 0.01$ s.

independently. In Fig. 3 we show stability curves for two different values of Re where the amplitude of perturbation required to cause transition is plotted as a function of the length of the perturbed flow in pipe diameters. Perturbations with amplitudes below the curve did not cause transition and decayed as they propagated downstream. On the other hand, disturbances which had amplitudes above the curve gave rise to sustained disordered flow downstream which had the form of a localized puff [11] at Re = 2170 or a slug of turbulence at Re = 4000. As discussed elsewhere [13], the threshold is probabilistic so that a mean value can be estimated with a narrow well-defined width. These are denoted by the error bars in Fig. 3 which indicate the width of the experimentally determined probability distribution of the transition. Hence each data point on the graph was obtained from 40 rehearsals of the experiment.

The perturbation was injected for a prescribed time and, since the disturbed flow was advected at the mean speed of the pipe flow, an estimate of the spatial extent of the disturbed flow immediately downstream of the injection point was made. This spatial scale is denoted by length* (in pipe diameters) in Fig. 3, and scaling by the mean flow collapses the two sets of data. Moreover, the amplitude of perturbation required for transition is independent of its length when more than six pipe diameters are initially disturbed. Shorter length perturbations result in a nonlinear response, and these results are in accord with previous work [13] where a short triangular form perturbation was used. These new results indicate extracting exponents from the previous data [13] is not straightforward.

We reinforce the above scaling argument with the results shown in Fig. 4 which contain data in the range $2000 \leq \mathrm{Re} \leq 5500$ where a 95% fully developed flow was achieved with the shorter pipe. Three stability curves are presented, and these are now discussed in turn. In the first, a short, 0.2 s, duration pulse was used, and this corresponds to a disturbance length of one pipe diameter at Re = 2000. This exhibits nonlinear behavior such that a rapidly increasing perturbation amplitude is required to cause transition as Re = 2000 is approached. In the next, a long duration (1.8 s) pulse was used, and this corresponds to perturbing nine pipe diameters at Re = 2000. Here, the independence of the amplitude of perturbation required to cause the transition discussed in connection with Fig. 3 for lengths $\geq 6D$ (diameters) is confirmed. In this case, independence has been uncovered over a range of Re. The final data set was obtained by varying the duration of the pulse in proportion to the mean flow such that the length of the flow field which was initially disturbed was kept constant at 2.25 pipe diameters. This corresponds to a perturbation of width 0.45 s at

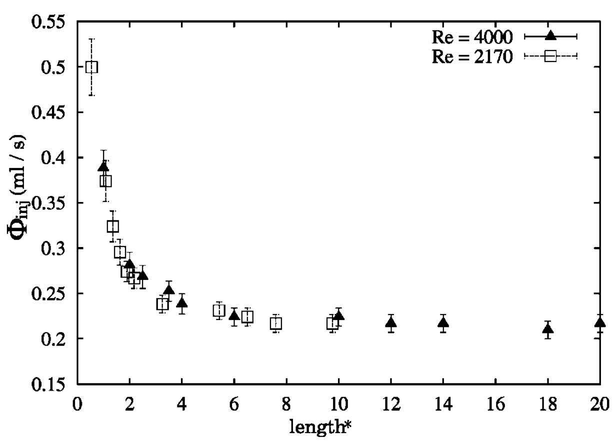

FIG. 3. Stability curves measured for Re = 2170 and Re = 4000. Each data point was obtained from 40 runs of the experiment, and the error bars correspond to the widths of the determined probability distributions [13]. The abscissa is the "length*" of the initially disturbed flow in pipe diameters.

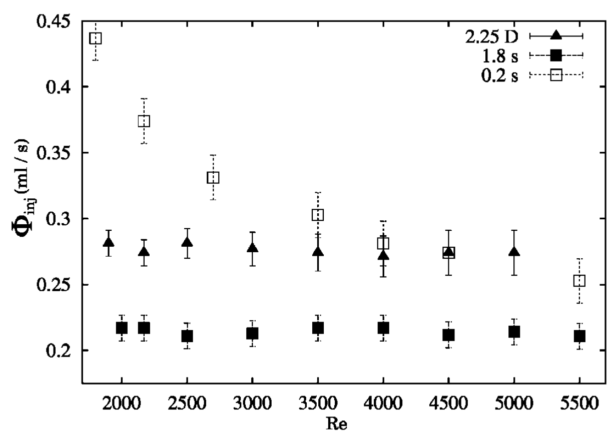

FIG. 4. Stability curves measured in the "short" pipe in the range $2000 \leq \mathrm{Re} \leq 5500$. Fixed duration perturbations were used to obtain the loci labeled 0.2 s and 1.8 s, respectively. The data set labeled $2.25D$ was measured using a variable duration pulse such that 2.25 diameters of the pipe flow were perturbed.

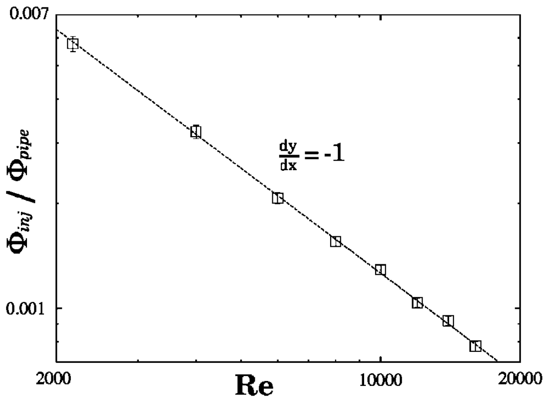

FIG. 5. A log-log plot of the stability curve obtained using the long pipe. The range of Re covered is 2000 to 18 000 and the amplitude of the perturbation has been nondimensionalized by the respective mass flux in the pipe. The least squares fitted line has a slope of -1 ± 0.01.

Re = 2000. The constant critical amplitude observed reinforces that scaling of the perturbation by the mean flow is valid.

It is clear that the two constant level thresholds in Fig. 4 cannot continue much below Re = 2000 since experimental evidence suggests that turbulent flow cannot be maintained below this value [11,13]. It is equally unlikely that the horizontal loci in Fig. 4 will simply come to an end in parameter space. In this region, we observe the transient growth of puffs which can persist for many tens of pipe diameters. This interesting behavior will take considerable experimental effort to resolve and is the subject of an ongoing investigation.

An appropriate scaling of the amplitude of the perturbation is the relative mass flux of the perturbation to that in the pipe. Clearly, doing this for the two horizontal loci in Fig. 4 will produce a proportionality of the form $O(\mathrm{Re}^{-1})$. We next present results from the long pipe in Fig. 5 where we were able to test this finding over an order of magnitude range of Re. Here we used a perturbation of 1.8 s duration and find the same $O(\mathrm{Re}^{-1\pm0.01})$ scaling. Obtaining this set of results was particularly challenging at the higher range of Re, and they required extremely tight control on background influences in our long pipe facility.

Clear experimental evidence for the scaling of the finite amplitude of perturbation required to promote transition in Poiseuille flow has been found. The exponent is -1 and has been uncovered using considerable care in the design and execution of the experiment. Interestingly, this exponent has also been found in experiments on transition in boundary layers [21]. Moreover, it is in agreement with recent asymptotic estimates for pipe flow [22] where transient growth plays a role. The exponent also indicates a generic transition [7] so that a challenge to theory is to provide a more definite indicator which will permit a distinction between competing ideas to be made.

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