Pseudo Angular Momentum and Topological Charge Conservation for Nonlinear Acoustical Vortices

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We study acoustical vortices (AVs) and their connection with optical vortices (OVs). We show that AVs and OVs have the same properties if the concept of pseudomomentum is used. Radiation stress and conservation of topological charge are obtained with the pseudomomentum rule. In a weak nonlinear regime, the conservation of pseudo angular momentum imposes a linear increase of the topological charge with the harmonic order. This last result is experimentally verified.

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Wave front dislocation is one of the three kinds of wave singularities [1]. These dislocations were first studied in acoustics and classified into two categories: edge and screw dislocations [2]. Afterward, this second type of dislocation was intensely studied in optics and named optical vortices (OVs). A significant stage was the discovery that these waves possess a quantified angular momentum proportional to the topological charge [3] and its experimental confirmation [4]. These singular waves have many analogies with vortices in superfluid. This interest was reinforced by the experimental realization of these waves and their many potential applications, such as precision alignment or optical spanners [4,5]. However, although dislocations were first studied in acoustics, there is only one relatively recent study done on acoustical vortices (AVs) [6].

In the present Letter, an analysis taking into account the specific nonlinearities induced by the existence of a reference frame, namely the material medium supporting the acoustic wave, is presented. This peculiarity requires one to distinguish the concepts of total momentum and pseudomomentum [7] and, as we will show, is also relevant for OVs propagating in a dielectric medium. We will derive a complete analogy between the pseudo angular momentum of a longitudinal acoustic wave and the one of a linearly polarized optical wave. This analysis will then be used to demonstrate that charge conservation of OV and AV propagating in heterogeneous media is related to pseudo angular momentum conservation. Although pseudomomentum involves only linear fields, the propagation of an acoustic wave is always a nonlinear phenomenon. These nonlinearities induce the generation of higher harmonics. It will be shown that the sum over all the harmonics of pseudo angular momentum and pseudoenergy will be conserved if the charge increases linearly with the order of the harmonic. Again this demonstration is also valid for OVs propagating in a nonlinear medium. Thereafter an original experimental setup is described. AVs of charges 1 and 2 are achieved at weak and finite amplitudes. The increase of the topological charge during the transfer of energy towards the harmonics is observed as well as the dislocation of higher order vortices in elementary vortices of charge 1. Finally, a short discussion demonstrates for radiation stress applications that the concept of angular pseudomomentum is also the relevant one.

The computation of the momentum of an acoustic wave is a tricky exercise because it requires the development of the mean field at second order. Indeed, the existence of a reference frame introduces nonlinear terms into the Euler equation and the equation of mass conservation. As an example, the mean velocity in Lagrange coordinates, the material reference frame at rest, is equal to the mean velocity in Euler coordinates, the instantaneous positions, at first order only [8]:

$$\langle v\rangle^L - \langle v\rangle = \rho_0^{-1} c_0^{-2}\langle I\rangle, \tag{1}$$

where $\langle v\rangle^L$, $\langle v\rangle$, $\langle I\rangle$, ρ_0, and c_0 are, respectively, the mean velocity in Lagrange and Euler coordinates, the wave energy density flux, the density, and the speed of sound of the medium at rest. Pressure and velocity at second order for a medium at rest may be written

$$v = v_1 + v_2, \qquad \text{with } \langle v_1\rangle = 0, \tag{2}$$

$$p = p_0 + p_1 + p_2, \qquad \text{with } \langle p_1\rangle = 0. \tag{3}$$

With these notations, Eq. (1) can be rewritten $I = p_1 v_1$

$$\langle v_2\rangle^L - \langle v_2\rangle = \rho_0^{-1} c_0^{-2}\langle p_1 v_1\rangle. \tag{4}$$

Thus, the difference between the average values, the Stokes drift, is proportional to the Poynting vector of the acoustic wave and involves linear fields only, whereas the total momentum in Euler coordinates is equal to

$$\langle \rho v\rangle = \langle \rho_1 v_1\rangle + \rho_0\langle v_2\rangle. \tag{5}$$

However, for a lossless medium the mean Lagrangian velocity is null so that

$$\rho_0\langle v_2\rangle^L = 0. \tag{6}$$

This condition neglects the acoustic streaming resulting from transfer of momentum in an absorbing medium [8].

Equations (4) and (6) give

$$\rho_0\langle v_2\rangle = -c_0^{-2}\langle p_1 v_1\rangle, \tag{7}$$

and, combined with the equation of state at first order $p_1 = c_0^2\rho_1$, one obtains

$$\rho_0\langle v_2\rangle = -\langle \rho_1 v_1\rangle. \tag{8}$$

This last equation shows that the 2 s order contributions cancel each other in a lossless medium. One can nevertheless define a pseudomomentum $g = \langle \rho_1 v_1\rangle = c_0^{-2}\langle I\rangle$. This pseudomomentum is defined with linear fields only. This last equation is also valid in optics. Indeed, the pseudomomentum, in Minkowski form, is equal to [9]

$$\frac{n}{c}U = \frac{n^2}{c^2}\langle E \wedge H\rangle = \frac{n^2\langle I\rangle}{c^2}, \tag{9}$$

where U is the density of energy of the electromagnetic wave, n is the index of the medium, and c is the speed of light in vacuum. This pseudomomentum should not be confused with the Abraham form which in certain cases is equal to the momentum of the wave [9]:

$$\frac{1}{nc}U = \frac{1}{c^2}\langle E \wedge H\rangle = \frac{\langle I\rangle}{c^2}. \tag{10}$$

The acoustic waves being always propagated in a medium, there is no equivalent of this last quantity in acoustics. Thereafter, this analogy between acoustical and optical pseudomomentum will be used to derive in acoustics the results previously obtained in optics. The pressure and velocity at first order of a longitudinal wave may be computed from a scalar potential:

$$p_1 = \rho_0\dot{\phi}, \qquad \text{and } v_1 = -\nabla\phi, \tag{11}$$

where the dot signifies the time derivative, so that the pseudomomentum reads

$$g = -\frac{\rho_0}{2c_0^2}\langle \dot{\phi}^*\nabla\phi + \dot{\phi}\nabla\phi^*\rangle, \tag{12}$$

where the star stands for phase conjugation. If one takes for ϕ the expression used by Allen *et al.* [3] for the potential vector of a linearly polarized optical wave,

$$\phi = u(x, y, z)e^{-i(kz-\omega t)}, \tag{13}$$

one obtains in acoustics

$$g = i\omega\frac{\rho_0}{2c_0^2}(u^*\nabla u - u\nabla u^*) + \omega k\frac{\rho_0}{c_0^2}|u|^2. \tag{14}$$

Hence, the acoustical pseudomomentum of a Gauss-Laguerre beam is identical to the optical momentum found by Allen *et al.* [3] with the vacuum permittivity, ε_0, replaced by $c_0^{-2}\rho_0$. However, Allen *et al.* used the Abraham expression, Eq. (10). A better analogy is obtained if one uses the Minkowski expression, Eq. (9), i.e., the optical pseudomomentum. In this case, instead of the permittivity of the vacuum, we have the permittivity of the dielectric material so that both acoustical and optical pseudomomentum will have the same conservation law. This last point is discussed in the next paragraph. Thus, the derivation detailed in Allen may be applied to the acoustics case and shows that an AV has the same pseudo angular momentum as an OV [10]:

$$M_z = l\omega\rho_0 c_0^{-2}|u|^2, \tag{15}$$

where l is the topological charge. Moreover, the relation between the Poynting vector and the pseudomomentum is the same one in optics and in acoustics so that the ratio between the angular and the linear pseudomomentum is

$$\frac{M_z}{g_z} = \frac{lc_0}{\omega}. \tag{16}$$

In optics, an additional term appears if the wave is circularly polarized [3]. This term does not exist for acoustic waves in a fluid which are longitudinal waves. But this term should be taken into account for acoustical transverse waves in a solid which may be circularly polarized.

The conservation of angular momentum is usually invoked to explain the conservation of the topological charge. Nevertheless, the conserved quantity is the total angular momentum, which includes not only the momentum of the electromagnetic wave but also the angular momentum of the atoms of the dielectric medium [9]. Indeed, the electromagnetic wave exerts forces on the atoms of the medium which acquires an angular momentum as long as the wave is present. This momentum part carried by the matter is excluded from the Abraham form. Furthermore, conservation of total momentum is related to the space isotropy and does not depend on the isotropy of the dielectric medium. However, cylindrical lens are used to transform the Hermitte-Gauss beam, which does not possess any topological charge, in OV [3]. More recently, cylindrical lenses were also used to reverse the topological charge of an optical vortex [11]. On the contrary, spherical lenses preserve the charge. These experiments show that the conservation of the topological charge is affected by the symmetry of rotation of the medium and, hence, follows the conservation law of pseudomomentum, i.e., Minkowski form. Indeed the conservation of the pseudomomentum is related to invariance by translation of the wave keeping the medium fixed, i.e., the medium's homogeneity, and the conservation of the pseudo angular momentum is related to invariance by rotation of the medium, i.e., its isotropy [7,9]. Thus, the relevant quantity in optics to study the topological charge is as in acoustics the pseudo angular momentum.

For the weak nonlinear regime, energy is transferred to harmonics of higher order of index i. The discussion presented above shows that the pseudo angular momentum should be conserved in an isotropic, lossless medium. Equation (15) may be written

$$\sum_i M_{iz} = \sum_i \frac{l_i}{\omega_i}\langle E_i \rangle, \tag{17}$$

where $\langle E_i \rangle$ is the pseudoenergy. Here we neglect the smaller term introduced by the helical structure of the wave front and assume a quasiplane wave front so that

$$\langle E_i \rangle = \frac{|p_i|^2}{2\rho_0 c_0^2} + \rho_0 \frac{|v_i|^2}{2} \approx \frac{|p_i|^2}{\rho_0 c_0^2} = \omega_i^2 \frac{\rho_0 |u_i|^2}{c_0^2}.$$

Equation (17) demonstrates that the pseudo angular momentum and the pseudoenergy are both conserved if and only if the ratio between the topological charge and the frequency does not depend on the harmonic order.

The conservation of the pseudo angular momentum in a nonlinear regime was experimentally checked. We synthesized acoustic vortices by using a network of 55 piezoelectric transducers (Imasonic, France) immersed in water. The transducers are set up on a flat surface with a hexagonal pattern. The array pitch is 11 mm and each element is a disk of 9 mm diameter. The central frequency of the elements is 1 MHz corresponding to a wavelength of 1.5 mm. The signals emitted by each piezoelectric element are computed with the inverse filtering technique [12]. The vortex distribution of phase and amplitude is set on an aperture of 80 mm by 80 mm located 400 mm away from the array. This plane is sampled with a step of 1.5 mm by a polyvinylidene fluoride hydrophone of 1 mm in diameter. The multichannel electronics that control the transducer array has a dynamics of 10 bit for a sampling rate of 80 MHz (Lecoeur Electronique, France). Each of the 128 channels is a generator of arbitrary signals that can deliver an average power of 25 W. The cross channel jitter is less than 12 ns. The amplitude and phase distribution is given by a Gauss-Laguerre beam with a waist of 22.5 mm. Two vortices are successively synthesized with charges 1 and 2, respectively. The time dependence is a burst of 25 μs at 1 MHz. After emission of the signals computed by inverse filtering, the phase and the average intensity of the pressure field are measured in the preset plane and displayed at 1 MHz (Fig. 1) for the vortex of charge 1 and 2. Linear variation of the phase around the axis of the vortex as well as the phase dislocation are nicely reproduced. The distribution of amplitude is also close to the objective, and the size of the vortex core is here of only two wavelengths for the vortex of charge 1. This sharpness is due to the very good spatial sampling of the array and as a corollary the very broad spatial bandwidth compared to computed holograms [13] or spatial light modulator [14].

These measurements are made again but for transmitted signals 10 times stronger than in the preceding experiment, so that now the nonlinear effects are significant and result in a transfer of energy toward the harmonics. To see how this phenomenon affects the AVs, the phase measurements are separately displayed for the second, fourth, and sixth harmonics of the vortex of charge 1 and the first, second, and third harmonics of the vortex of charge 2 (Fig. 2). The nonlinearity for a longitudinal acoustic wave is quadratic and one thus obtains all the multiples of the fundamental frequency. It is clearly observed that the topological charge is not preserved and increases linearly with the order of the harmonic. Hence, a vortex of charge 1 generates by parametric interaction all the vortices of higher order while a vortex of charge 2 will only give rise to vortices of even charge. Indeed the condition of synchronism for the transfer of energy towards the harmonics imposes that the wave vectors are collinear. Consequently, the wave front is also helical with the same step height for all the harmonics [10]. This experiment also confirms that higher order AVs are unstable and break up into elementary vortices of

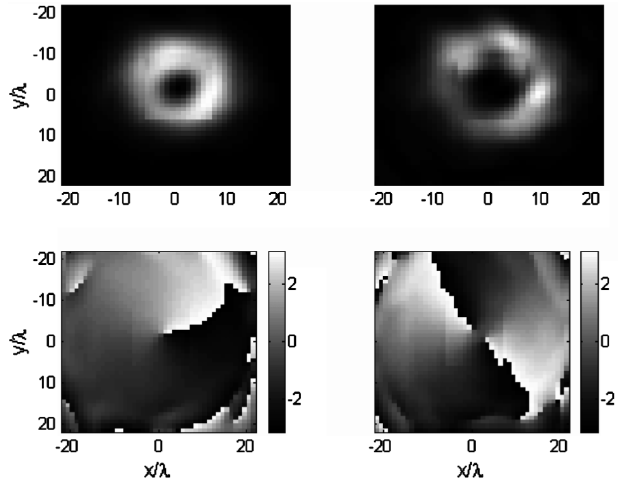

FIG. 1. Average intensity (top figures) and instantaneous phase (bottom figures) measurements of AVs of charges 1 (left) and 2 (right) at weak amplitude.

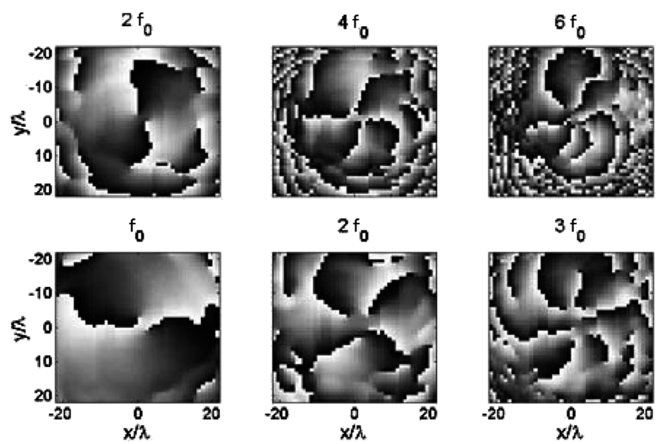

FIG. 2. Instantaneous phase measurements of harmonics 2, 4, and 6 of the vortex of charge 1 (top figures) and of harmonics 1, 2, and 3 of the vortex of charge 2 (bottom figures) at finite amplitude.

charge 1 [13]. This instability takes place, on the one hand, on the vortex of charge 2 synthesized linearly by inverse filtering, and, on the other hand, on the vortices of higher order generated by parametric interaction. Nevertheless the vortex of charge 6 generated by parametric interaction from the vortex of charge 1 is definitely less spread out than the vortex with the same charge but resulting from the vortex of charge 2. Indeed, harmonics are gradually amplified by energy conversion from lower frequency that tends to counterbalance the spreading.

Another important feature of OV is the associated radiation stress for applications such as optical spanners. Here again for either AV or OV the relevant quantity is the pseudomomentum [7,15]. More precisely in acoustics, this connection comes from the close relationship between the flux of pseudomomentum and the flux of momentum. In fluids, the difference between these two tensors, if any, comes from the isotropic pressurelike contribution proportional to the coefficient of nonlinearity of the medium. For instance, this contribution explains the difference between the Rayleigh and the Langevin radiation stress [7]. However, the torque is computed from the nondiagonal part of the flux of momentum tensor so that the pseudomomentum rule always holds: One may consider that the medium is absent and the wave carries an angular momentum equal to its pseudo angular momentum or, in other words, that the angular momentum is $l\hbar$ per phonon. Thus, a wave vortex will exert a mechanical torque proportional to its pseudo angular momentum variation. A mechanical torque will be exerted either on an absorbing medium or on a medium whose property varies with the azimuthal angle.

In conclusion, we introduced the concept of pseudo angular momentum for both OVs and AVs. This analysis emphasizes the distinction between total and pseudomomentum when the wave propagates in a material medium. Thus, in the linear regime the topological charge of a wave vortex is associated with the pseudo angular momentum conservation resulting from the medium isotropy. For the weak nonlinear regime, conservation of pseudoenergy and pseudo angular momentum is satisfied if the ratio between the charge and the frequency remains constant. Moreover, the torque exerted on an absorbing or anisotropic medium may ever be computed assuming that the wave carries an angular momentum equal to its pseudo angular momentum. The experimental setup presented here is very versatile and the AV spatial and temporal patterns are configurable in real time. This characteristic is of potential interest for trapping and manipulation applications. Another potential application is the synthesis of dark acoustical solitons; see [16] for a review about dark optical solitons. Indeed, at second order the speed of sound increases linearly with the wave amplitude so that the nonlinearity is defocusing.

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