# Experimental Observation of Multifrequency Patterns in Arrays of Coupled Nonlinear Oscillators

Visarath In,[1,*] Andy Kho,[1,†] Joseph D. Neff,[1,‡] Antonio Palacios,[2,§] Patrick Longhini,[2,||] and Brian K. Meadows[1,¶]

[1]Space and Naval Warfare Systems Center, Code 2363, 53560 Hull Street, San Diego, California 92152-5001, USA
[2]Nonlinear Dynamics Group, Department of Mathematics & Statistics, San Diego State University, San Diego, California 92182, USA

(Received 5 February 2003; published 11 December 2003)

Frequency-related oscillations in coupled oscillator systems, in which one or more oscillators oscillate at different frequencies than the other oscillators, have been studied using group theoretical methods by Armbruster and Chossat [Phys. Lett. A **254**, 269 (1999)] and more recently by Golubitsky and Stewart [in *Geometry, Mechanics, and Dynamics*, edited by P. Newton, P. Holmes, and A. Weinstein (Springer, New York, 2002), p. 243]. We demonstrate, experimentally, via electronic circuits, the existence of frequency-related oscillations in a network of two arrays of $N$ oscillators, per array, coupled to one another. Under certain conditions, one of the arrays can be induced to oscillate at $N$ times the frequency of the other array. This type of behavior is different from the one observed in a driven system because it is dictated mainly by the symmetry of the coupled system.

DOI: 10.1103/PhysRevLett.91.244101 PACS numbers: 05.45.Xt

Theoretical works in coupled oscillators have led to the observation of peculiar patterns of oscillations, where one or more oscillators oscillate at different frequencies [1–4]. We refer to such patterns as "multifrequency oscillations." Experimentally, Takamatsu *et al.* [5] reported a wide variety of oscillatory patterns in rings of coupled biological oscillators of the plasmodial slime mold, including a multifrequency pattern, described as "a novel oscillation mode," where one oscillator of a three-oscillator ring oscillates at half the period (or twice the frequency) of the other two. In this work, we now show that multifrequency patterns can be realized in an electronic circuit that serves as a model for a network consisting of two coupled arrays of $N$ oscillators per array. Then we demonstrate a particular pattern where in-phase oscillators in one array are induced to oscillate at $N$ times the frequency of the opposite array. Additionally, we elucidate possible network symmetries that can lead to similar behavior but when the oscillators are out of phase by a constant amount, i.e., a traveling wave multifrequency pattern. This type of pattern is of great importance for certain engineering applications, such as in the design of antenna and radar systems, for the following reasons. Many antenna and radar systems are made up of arrays of coupled nonlinear oscillators, where each individual oscillator has its own natural frequency of oscillation and power output. The oscillators are normally coupled in an array structure to produce a collective pattern of oscillation with a common frequency and enhanced power output. Under current technology [6], it is possible to manipulate operational parameters (coupling strength and topology of connections) to steer and scan the beam that emanates from the oscillators. Changing the collective frequency, however, is not yet possible. But it is our hope that this work can lead to possible alternatives.

Central to the works mentioned above is the use of symmetry in a systematic way. By systematic we mean the group theoretical approach developed by Golubitsky and co-workers [2,7–9] to study symmetric systems. Such an approach is model independent because the results are dictated exclusively by the symmetry of the system regardless of the nature of the oscillators. In this work, we also use symmetry in a systematic way to identify certain multifrequency patterns that, otherwise, would be difficult to find through the standard theory of synchronization or frequency entrainment [10]. The Letter is organized as follows. First, we argue that certain spatiotemporal symmetries can induce one array of a two-array coupled network to oscillate at $N$ times the frequency of the other, where $N$ is the number of oscillators in each array. Then we present an electronic version of the network constructed with overdamped Duffing oscillators, which demonstrates good agreement with the theory.

Following Buono *et al.* [3] and related works [2,11–13], we assume that the internal dynamics of each individual oscillator is governed by a $k$-dimensional continuous-time system of differential equations of the form

$$\frac{dX_i}{dt} = f(X_i, \lambda), \tag{1}$$

where $X_i = (x_{i1}, \ldots, x_{ik}) \in \mathbf{R}^k$ denotes the state variables of oscillator $i$ and $\lambda$ is a vector of parameters. Observe that $f$ is independent of $i$ because the oscillators are assumed to be identical. A network of $N$ oscillators is a collection of $N$ identical oscillators, interconnected in some fashion, which can be modeled by the following system of coupled differential equations:

$$\frac{dX_i}{dt} = f(X_i) + \sum_{j \to i} c_{ij} h(X_i, X_j), \tag{2}$$

where $h$ is the coupling function between two oscillators,

 0031-9007/03/91(24)/244101(4)$20.00 © 2003 The American Physical Society 

the summation is taken over those oscillators $j$ that are coupled to oscillator $i$, and $c_{ij}$ is a matrix of coupling strengths.

Figure 1 illustrates the network under consideration for this work. To study its collective behavior, we use $X(t) = (X_1(t), \ldots, X_N(t))$ to represent the state of the oscillators on the left-hand-side array, and $Y(t) = (Y_1(t), \ldots, Y_N(t))$ for that of the oscillators on the opposite-side array. At any given time, the state of the entire network is described by $(X(t), Y(t))$. Of particular interest to this work are traveling wave (TW) patterns (i.e., periodic oscillations of period $T$ with identical wave form but with a constant phase shift, $\phi = T/N$, among nearest neighboring oscillators) and in-phase (IP) oscillations of the same period $T$. For instance, $P_1(t) = (X_{\mathrm{TW}}(t), Y_{\mathrm{IP}}(t))$ describes a collective pattern where the left-hand-side array oscillates in a TW fashion, $X_{\mathrm{TW}}(t) = [X_1(t), X_1(t-\phi), \ldots, X_N(t-(N-1)\phi)]$, while the opposite array oscillates in phase, $Y_{\mathrm{IP}}(t) = (Y_1(t), \ldots, Y_N(t))$, and $Y_i(t) = Y_j(t)$.

The amount of symmetry of these and other similar patterns is described by the set of spatial and temporal transformations that leave them unchanged. Together, these transformations form the group of symmetries of the pattern. Then a critical observation is the fact that certain groups of symmetries are associated with periodic patterns where an entire array oscillates at different frequencies. For instance, assume that the pattern $P_1 = (X_{\mathrm{TW}}(t), Y_{\mathrm{IP}}(t))$ has symmetry group $(\mathbf{Z}_N \times \mathbf{S}^1) \times (\mathbf{Z}_N \times \mathbf{S}^1)$, which describes (simultaneous) cyclic permutations of the oscillators in each array, accompanied by shifts in time by $\phi$. Then a direct exercise [2] shows that these two operations leave the traveling wave unchanged, but the in-phase oscillators are shifted in time by $\phi$. That is, $(\mathbf{Z}_n \times \mathbf{S}^1) \cdot X_{\mathrm{TW}}(t) = X_{\mathrm{TW}}(t)$ and $(\mathbf{Z}_n \times \mathbf{S}^1) \cdot Y_{\mathrm{IP}}(t) = Y_{\mathrm{IP}}(t+\phi)$. Thus if $(\mathbf{Z}_N \times \mathbf{S}^1) \times (\mathbf{Z}_N \times \mathbf{S}^1)$ is the symmetry group of $P_1(t)$, then $(\mathbf{Z}_n \times \mathbf{S}^1) \times (\mathbf{Z}_N \times \mathbf{S}^1) \cdot P_1(t) = P_1(t)$, which implies that $Y_{\mathrm{IP}}(t) = Y_{\mathrm{IP}}(t+\phi)$. In simpler words, the in-phase pattern must oscillate at $N$ times the frequency of the traveling wave pattern.

Next we illustrate this previous conclusion with a system of coupled overdamped Duffing oscillators, using both numerical simulations (similar to those used by Golubitsky and Stewart [2]) and measurements from electronic circuits. Two different interconnection schemes are considered separately, depending on whether $N$ is odd or even. In the odd case, the oscillators of each array are unidirectionally coupled to their nearest neighbors. When $N$ is even, however, non-nearest neighbor oscillators are additionally coupled to one another in order to meet the necessary conditions for the Hopf bifurcations that lead the arrays to oscillate [4] and, consequently, for the multifrequency pattern to be observed. In either case, the arrays are then interconnected to one another via sums of outputs — as is shown in Fig. 1. The resulting network can then be modeled by a system of differential equations of the form

$$\begin{aligned} \dot{x}_i &= \lambda_x x_i - x_i^3 + c_x(x_i - x_{i+1}) + c_{xy}\sum_{i=1}^{N} y_i, \\ \dot{y}_i &= \lambda_y y_i - y_i^3 + c_y(y_i - y_{i+1}) + c_{xy}\sum_{i=1}^{N} x_i, \end{aligned} \tag{3}$$

where $x_i$ ($y_i$) are the state variables of the left-hand (right-hand) arrays, $\lambda_x$ ($\lambda_y$) is a parameter that controls the local dynamics of each oscillator in the left-hand (right-hand) arrays, $c_x$ ($c_y$) is the coupling strength for connections in the $X$ ($Y$) array, and $c_{xy}$ is the cross-coupling strength between the $X$ and $Y$ array. A linear stability analysis reveals that each array is capable of oscillating on its own, i.e., when $c_{xy} = 0$, via a Hopf bifurcation at $\lambda_x = -(3/2)c_x$ or $\lambda_y = -(3/2)c_y$. Figure 2 shows the results of integrating the model Eqs. (3) with $N = 3$ oscillators per array. Initial conditions and parameters are $X_0 = (1.78, -0.85, -0.08)$, $Y_0 = (0.99, 0.99, 0.99)$, $c_x = c_y = 2.6$, $c_{xy} = 0.01$, and $\lambda_x = \lambda_y = 0.02$. As predicted by theory, the in-phase oscillations are $N = 3$ times faster

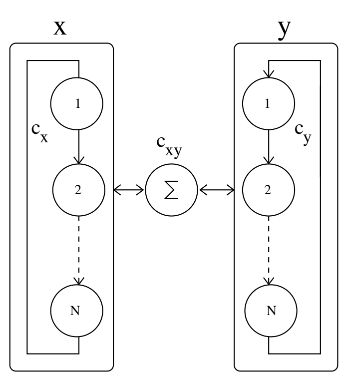


FIG. 1. Schematic diagram of a two-array network of coupled oscillators with $N$ oscillators per array.

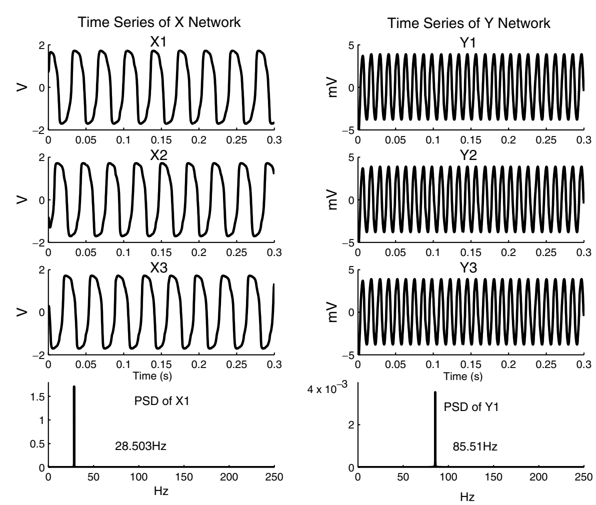


FIG. 2. Numerical simulations of the network Eqs. (3). The frequency spectrum (bottom graph) confirms that the in-phase wave of the $Y$ array oscillates at 3 times the frequency of the traveling wave of the $X$ array.

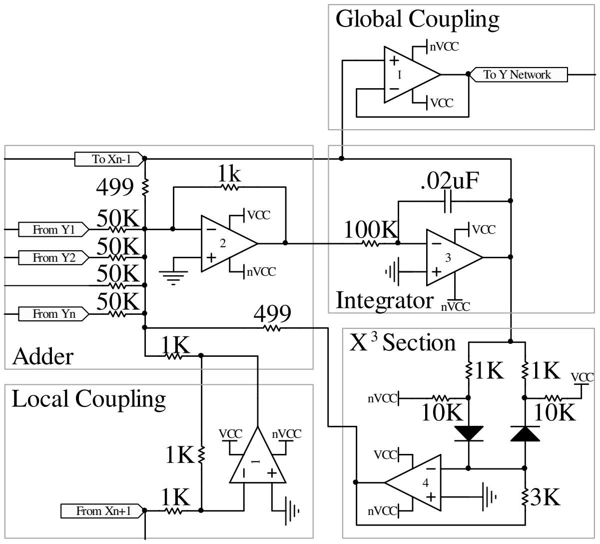


FIG. 3. Electronic circuit diagram of a unit oscillator of the overdamped Duffing oscillator arrays in Fig. 4. All circuit components and values are listed in the diagram. Op-amps are LM348N made by National Semiconductor.

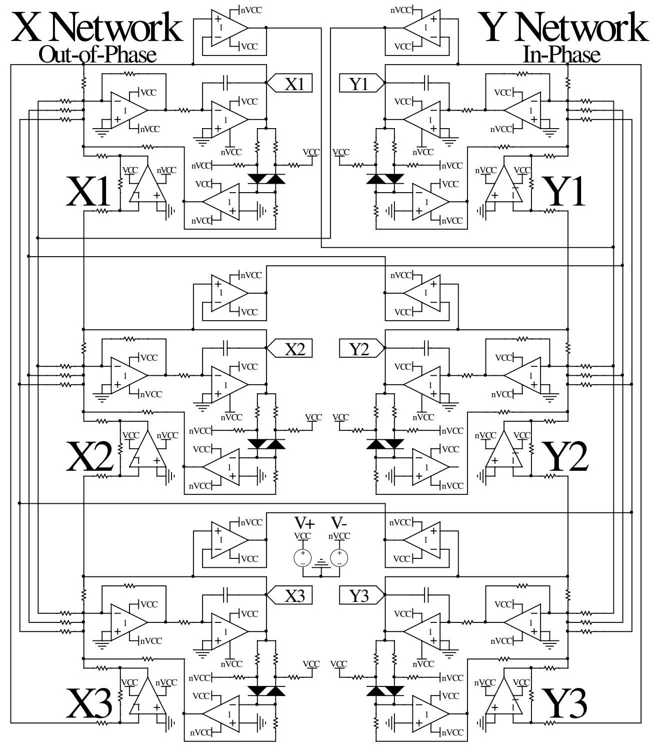


FIG. 4. Electronic analog of the network of two coupled arrays shown in Fig. 1. To guard against parasitic coupling between the arrays, each array is provided with its own power regulators.

than those of the traveling wave. Further simulations with larger arrays ($N > 3$) produce similar multifrequency patterns.

An experimental analog of Eq. (3) was constructed using electronic components. The circuit consists mainly of operational amplifiers (op-amps), which act as summing-inverting amplifiers, integrators, and produce the linear and nonlinear terms; these functions represent the local dynamics of a unit oscillator; see Fig. 3. Additional op-amps are used to do the unidirectional coupling between oscillators in an array and the cross-coupling terms between the arrays. All values of the components used in the circuit are given in Fig. 3. In the electronic network, the arrays consist of three unit oscillators, forming two $1 \times 3$ columns representing the $X$ and $Y$ arrays, respectively; see Fig. 4. Although attempts were made to match the parameters between the experimental and numerical systems, they differ due to the availability of parts and to the difficulty in producing the nonlinear term, which in the electronic circuit is implemented as a piecewise linear function. However, it is not necessary to match the parameters in order to achieve similar behavior since the resulting pattern is dependent on the type of coupling and the symmetry of the system.

When the circuit is first powered up, both the $X$ array and the $Y$ array tend to oscillate at the same frequency, but both are in a traveling wave state; i.e., each oscillator in each array is shifted in phase from its neighbor by $T/3$, where $T$ is the mutual period of oscillation. This pattern represents one of the many possible solutions that can be found merely by investigating the symmetry properties of the system. This particular solution is apparently favored as a result of the power-up initial conditions, i.e., in its basin of attraction. To get only one array to oscillate in phase, such as the $Y$ array, the initial conditions or state of the array need to be changed. This is done by briefly pinning the voltage at $Y1$, $Y2$, or $Y3$ with a separate power supply; see Fig. 4. In the experiment, $Y1$ is momentarily set to 1.7 V. After a few tries the array can be induced into its in-phase state, after which the state is stable and self-sustaining.

In the experiment the nominal (traveling wave oscillations) frequency is approximately 28.44 Hz. When one of the arrays is induced to operate in the in-phase state, the oscillations in that array are 3 times (approximately 85.27 Hz) the nominal frequency; see Fig. 5. In this figure, the left column shows the traveling wave dynamics of oscillators 1, 2, and 3 in the $X$ array. The right column shows the dynamics of the corresponding oscillators in the $Y$ array. The two graphs at the bottom depict the power spectral density plots of the corresponding time series represented in each column. The bottom right graph clearly illustrates that the $Y$ array is oscillating, within the experimental error, at 3 times the oscillation of the $X$ array. As is the case in the numerical simulations, the multifrequency behavior can occur only within a certain window in parameter space.

To summarize, using symmetry-based arguments, we have shown that multifrequency patterns occur commonly in a model of coupled arrays of nonlinear oscillators. In particular, a two-array network with $N$

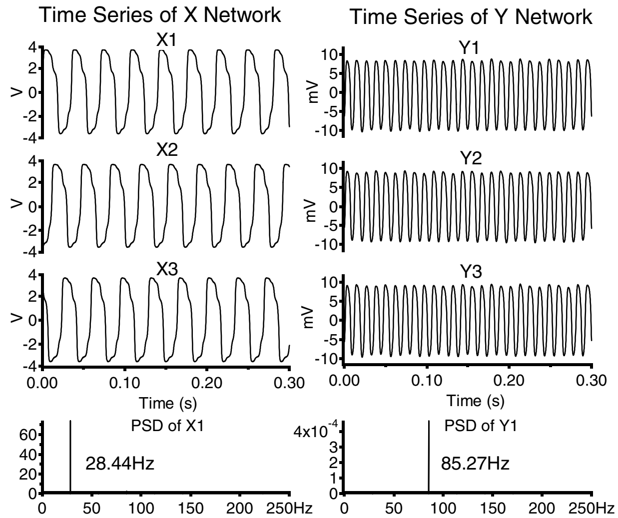


FIG. 5. Voltage measurements of the electronic network depicted in Fig. 1 further confirm the numerical results of Fig. 2.

identically coupled oscillators per array was shown to have one array oscillate in a traveling wave pattern while the other array oscillates in phase but at $N$ times the frequency of the traveling wave state. In principle, the number of elements in each array does not have to be identical; therefore, the frequency of the in-phase state can be changed based on the number of elements in the other array. This conclusion explains the novel mode of oscillation found previously by Takamatsu *et al.* in biological oscillators. Furthermore, a system of overdamped Duffing oscillators was used to illustrate the concepts with numerical simulations with $N = 3$ oscillators. An experimental analog of the system was also constructed with electronic circuits. Both the numerical simulations and the experimental data confirm the existence of multifrequency behavior in the system. We also emphasize that the model-independent feature of the symmetry methods used in this work imply that our results are valid for a general class of coupled oscillators regardless of the nature of the intrinsic dynamics of each oscillator. A second case of interest to this work is finding multifrequency oscillations in traveling wave patterns. This task is part of future work and incorporates the idea of using buffer oscillators between arrays of oscillators [14]. Buffer oscillators may lead to the desired behavior without the need for additive coupling between the arrays.

We are grateful to L. Buno and M. Golubitsky for informative discussions. Also, all the authors acknowledge the support from SPAWAR internal funding (ILIR) program.

*Electronic address: visarath@spawar.navy.mil
†Electronic address: kho@spawar.navy.mil
‡Electronic address: jdneff@spawar.navy.mil
§Electronic address: palacios@euler.ssu.edu
‖Electronic address: plonghini@netzero.net
¶Electronic address: bmeadows@spawar.navy.mil

[1] D. Armbruster and P. Chossat, Phys. Lett. A **254**, 269 (1999).
[2] M. Golubitsky and I. Stewart, in *Geometry, Mechanics, and Dynamics*, edited by P. Newton, P. Holmes, and A. Weinstein (Springer, New York, 2002), p. 243.
[3] P.-L. Buono, M. Golubitsky, and A. Palacios, Physica (Amsterdam) **143D**, 74 (2000).
[4] M. Golubitsky and I. Stewart, in *Hopf Bifurcation with Dihedral Group Symmetry*, edited by M. Golubitsky and J. Guckenheimer, AMS Contemporary Mathematics Vol. 56 (Springer, New York, 1986), p. 131.
[5] A. Takamatsu, R. Tanaka, H. Yamada, T. Nakagaki, T. Fujiii, and I. Endo, Phys. Rev. Lett. **87**, 078102 (2001).
[6] T. Heath, K. Wiesenfeld, and R. A. York, Int. J. Bifurcation Chaos Appl. Sci. Eng. **10**, 2619 (2000).
[7] M. Golubitsky and I. Stewart, *Symmetry and Pattern Formation in Coupled Cell Networks*, edited by M. Golubitsky, D. Luss, and S. H. Strogatz, IMA Volumes in Mathematics and Its Applications Vol. 115 (Springer, New York, 1999), p. 65.
[8] M. Golubitsky, I. N. Stewart, and D. G. Schaeffer, in *Singularities and Groups in Bifurcation Theory: Vol. II*, Appl. Math. Sci. Vol. 69 (Springer-Verlag, New York, 1988).
[9] M. Golubitsky, M. Nicol, and I. Stewart (to be published).
[10] A. Pikovsky, M. Rosenblum, and J. Kurths, in *Synchronization. A Universal Concept in Nonlinear Sciences*, Cambridge Nonlinear Science Series Vol. 12 (Cambridge University Press, Cambridge, United Kingdom, 2001).
[11] B. Dionne, M. Golubitsky, and I. Stewart, Nonlinearity **9**, 559 (1996).
[12] I. Epstein and M. Golubitsky, Chaos **3**, 1 (1995).
[13] M. Golubitsky, I. N. Stewart, P-L. Buono, and J. J. Collins, Physica (Amsterdam) **115D**, 56 (1998).
[14] Marty Golubitsky (personal communication).