Noise-Controlled Self-Replicating Patterns

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We present novel numerical evidence of spot self-replication controlled by noise in a simple autocatalytic reaction-diffusion system. The system dynamics exhibits a noise controlled transition from stripe growth to spot replication. The growth kinetics is also controlled by noise, and there is an optimal noise intensity for which the multiplication rate of spots is maximal. For larger noise intensities, the spots become unstable and the system is attracted by the trivial steady state. Some of the effects are reminiscent of both polymer chain and cell colony formation in random environments.

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It is widely accepted nowadays that noise can lead to a rich variety of dynamical effects [1,2]. Far from being merely a (small) perturbation to *idealized* deterministic behavior or regarded as an undesirable source of randomness and disorganization, noise can *induce* radical and counterintuitive dynamical changes. Especially noteworthy are systems in which noise has organizing rather than disruptive effects. Some of the more important examples are stochastic resonance [2] and noise-induced transitions [1].

Noise is pervasive in all realizable dynamical systems either as internal noise, for instance, thermal fluctuactions, or as external noise, e.g., due to environmental contingencies. In fact, the quantitative description of many complex systems is often simplified via coarse graining, yielding a macroscopic phenomenological model, where the fluctuations of the (decoupled) microscopic degrees of freedom are averaged out. However, the degrees of freedom thus treated induce dynamical effects in the macroscopic model that can be represented as internal noise [3]. The dynamics may also evolve in a medium which provides external perturbations and unpredictable disturbances. Therefore, it is important to consider the influence of external noise on phenomenological models and their dynamical evolution.

The elucidation of dynamic organization is of paramount importance in prebiotic chemistry and in biology, for the latter can be viewed as a paradigm of complexity and self-organization. The emergence of collective and organized behaviors is a widely studied subject [4], but the decisive role played by noise in self-organization is perhaps, to date, the least understood.

In his seminal paper [5], Turing showed that simple and deterministic reaction-diffusion systems can lead to a wide range of pattern forming instabilities. Since then, reaction-diffusion models have become a focus of intense study and an archetype of pattern formation in fields as diverse as biology, chemistry, applied mathematics, and engineering. Experiments in gel-filled reactors have revealed rotating spirals [6], stationary Turing patterns [7], waves [8], and self-replicating spots [9,10], among others. Self-replicating spots have also been observed in numerical simulations of the Gray-Scott model [11], the Fitzhugh-Nagumo model [12], the Garpar-Showalter model [10], and in others [13]. However, it is with Gray-Scott [14] that most of the studies of self-replicating patterns have been performed in one [15–19] and two dimensions [11,20]. For a detailed explanation of spot replication, see Fig. 3 of [9] and Fig. 4 of [11]. In addition to self-replicating patterns, the two-dimensional Gray-Scott model reveals a rich variety of spatiotemporal patterns, so that it is a suitable laboratory model on which to apply novel techniques to the pattern formation problem in reaction-diffusion models. Regarding the effects of noise in the Gray-Scott model, its large scale emergent properties have been recently studied by means of a renormalization group analysis [21].

In this Letter, a novel and interesting effect, the appearance of self-replicating patterns *controlled by noise* in a stochastic reaction-diffusion model, is described. Using a two-dimensional autocatalytic system with additive noise, we have performed numerical simulations of noise-induced transitions between different evolving morphologies. These studies raise important questions on the role of noise in chemical and biological self-organization and the environmental *selection* of emergent patterns.

Model and methodology.—The system analyzed here is a stochastic version of the two-dimensional Gray-Scott model defined, in terms of dimensionless units, by the following equations (η_v and η_u denote additive noise terms):

$$\begin{aligned} \frac{\partial}{\partial t}V &= D\nabla^2 V + UV^2 - k_v V + \eta_v, \\ \frac{\partial}{\partial t}U &= \nabla^2 U - UV^2 + k_u(1 - U) + \eta_u. \end{aligned} \tag{1}$$

In the absence of noise, Eq. (1) defines the Gray-Scott model, which is a variant of the autocatalytic Selkov model of glycolysis. The Gray-Scott model describes a

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simple autocatalytic reaction in an unstirred or gel-filled reactor. The autocatalyst V undergoes a cubic autocatalytic reaction (unit reaction constant) using the substance U as a substrate that is, in turn, fed by affinity from an external reservoir (feed rate k_u) with fixed unit concentration. The autocatalyst V is degraded to an inert product (constant k_v). That is,

$$\begin{aligned} \text{reservoir} &\xrightarrow{k_u} U, \\ U + 2V &\xrightarrow{1} 3V, \\ V &\xrightarrow{k_v} \text{inert}. \end{aligned} \tag{2}$$

The chemical species U and V diffuse with independent diffusion constants. The diffusion constant of U is normalized to one, so that D, the ratio between both diffusion constants, remains as the only diffusion parameter.

The effects of external perturbations are included in this phenomenological model as additive Gaussian white noise. That is, the random fluctuations $\eta_u(\mathbf{x}, t)$ and $\eta_v(\mathbf{x}, t)$ added to (1) are Gaussian white zero-mean noises; the only nonvanishing cumulants are

$$\begin{aligned} \langle \eta_v(\mathbf{x}, t)\eta_v(\mathbf{x}', t')\rangle &= 2\mathcal{A}_v \delta^2(\mathbf{x} - \mathbf{x}')\delta(t - t'), \\ \langle \eta_u(\mathbf{x}, t)\eta_u(\mathbf{x}', t')\rangle &= 2\mathcal{A}_u \delta^2(\mathbf{x} - \mathbf{x}')\delta(t - t'), \end{aligned} \tag{3}$$

and there are no crossed correlations.

For the numerical simulations, a finite system size, 560 by 560, with periodic boundary conditions, has been chosen. The work in [11] shows that a large variety of distinct patterns can be obtained by making small changes in just the two parameters k_u and k_v. Instead, we are interested in how noise affects the dynamics for fixed deterministic parameters and the extent to which noise is capable of changing the patterns exhibited by the deterministic system. To this end, we fix the deterministic model parameters to the values $k_u = 0.05$, $k_v = 0.1155$, and $D = 0.5$, since the stripe morphology is near a transition point and hence, sensitive to fluctuations. We perform a study of the effects of noise by varying the noise intensity $\mathcal{A}_u$ in a range from 10^{-6} up to 10^{-2}. $\mathcal{A}_v$ has been switched off after checking that there are no qualitative differences between perturbing the substrate concentration only or perturbing both concentrations.

The numerical simulations of system evolution have been performed using forward Euler integration of the finite-difference equations following discretization of space and time in the stochastic partial differential equations. The spatial mesh consists of a lattice of 256 by 256 points (cell size $\Delta x = 2.2$). We have found that this mesh size provides a decent balance between acceptable resolution and computation time. Noise has been discretized as well and simulated by means of a generator with Gaussian distribution. The initial conditions consisted of one localized square pulse perturbing the trivial steady state ($U = 1$, $V = 0$) plus random Gaussian noise. The perturbing pulse measured 22×22, just wide enough to allow the autocatalytic reaction to be locally self-sustaining. The system has been numerically integrated for up to 20 000 time steps (step size $\Delta t = 1$). Larger time steps of up to 200 000 were used in preliminary simulations, but the spots and stripes were observed to grow until saturation (slowing down of pattern growth) set in at around 50 000 steps.

Results.—In the numerical simulations carried out, we observed that slight changes in noise intensity induced *dramatic* changes in the dynamical behavior of the system. When the noise level increases, the prevailing growth pattern changes from elongated stripes to self-replicating spots (Fig. 1). For still higher noise intensity, a disordering transition takes place: spots become unstable and the system is attracted by the trivial homogeneous steady state ($U = 1$, $V = 0$) in equilibrium with the reservoir. In the figures, only the concentration of substrate U is shown. Red represents the trivial steady state ($U = 1$). Blue represents a concentration between 0.2 and 0.4, where the substrate is being depleted by the autocatalytic production of V. Yellow represents an intermediate concentration of roughly 0.8.

In the absence of noise, or for a small noise intensity, the system evolves forming stripes [Figs. 1(a)–1(d)]. Stripe growth takes places at the ends, in the direction parallel to the stripe. For yet higher noise intensity ($\mathcal{A}_u$ above 10^{-3}), the spots split or fission into two spots, well before they can form a stripe. The spots replicate in a process that is visually remarkably similar to cell reproduction [Figs. 1(e)–1(h),]. This particular pattern has also been observed in the deterministic Gray-Scott model although for a different range of system parameters [11].

Solitary stable spots in two dimensions are described in [20]. In the simulations, we have observed that, for the chosen parameters, solitary spots are stable and unable to grow or replicate in the absence of noise [Fig. 2(a)]. The spot has a round shape and there is no preferred direction in which the spot becomes elongated [22]. However, growth starts readily if the rotational symmetry is broken, either because the spot is not solitary or by adding random noise. Noise can trigger spot growth, as we have observed in the simulations. Even small random perturbations ($\mathcal{A}_u = 10^{-6}$) are sufficient to break the symmetry and make it unstable.

For low noise intensity ($\mathcal{A}_u < 10^{-4}$), the prevailing propagating pattern is stripe growth. However, very often, even for low noise intensity, growth begins with one or more spot replications, so that, at the end of the simulation, several stripes are found [Fig. 2(b)]. Stripe nucleation is facilitated when two spots grow touching back to back. Later on, all the growing spots nucleate stripes and, finally, the pattern evolves strictly by stripe growth (a process that reminds one of polymerization).

For intermediate noise intensities both patterns, stripes and spots, can be observed simultaneously. Some of the spots formed at the beginning may get blocked by stripes

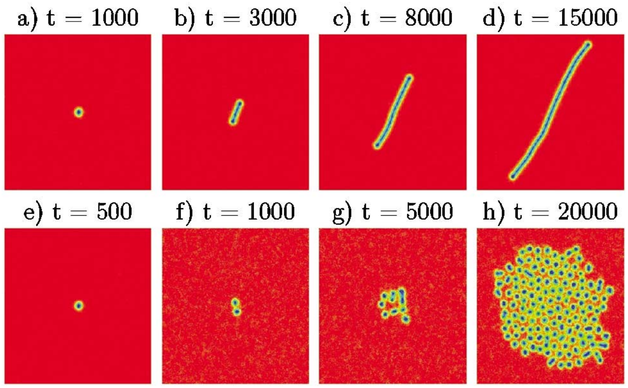

FIG. 1 (color). In the first row, elongated stripe growth for $\mathcal{A}_u = 6 \times 10^{-5}$. In the second row, self-replicating spots for $\mathcal{A}_u = 2 \times 10^{-3}$. The color represents the relative concentration of substrate U (red ≈ 1, yellow ≈ 0.8, and blue ≈ 0.3).

or other spots, so that they do not nucleate stripes and remain as spots. As a result, a hybrid pattern of stripes and spots appear [Fig. 2(c)] around $\mathcal{A}_u = 6 \times 10^{-5}$. The more intense the noise, the more difficult it is for the stripes to nucleate, and the more spots remain at the end of the calculation. For $\mathcal{A}_u$ greater than 4×10^{-4}, a new effect enters into play. Moderate perturbations can make stripe tips separate (Fig. 3), producing more spots and shorter stripes [Fig. 2(d)]. For more intense noise, greater than $\mathcal{A}_u = 10^{-3}$, stripes are sufficiently unstable that growing spots break up into two spots before forming stripes. Subsequent evolution is purely by spot replication [Fig. 2(e)]. So, the transition from stripe growing to spot replication can be summarized as due to two noisy effects: (i) Stripe nucleation becomes more difficult, and (ii) the "budding off" of spots from stripes due to tip instability is favored. Spots prove to be more robust to perturbations than stripes. The reason may be in the way the substrate is flowing to keep the autocatalytic reaction going. Whereas the spot is fed with the substrate by

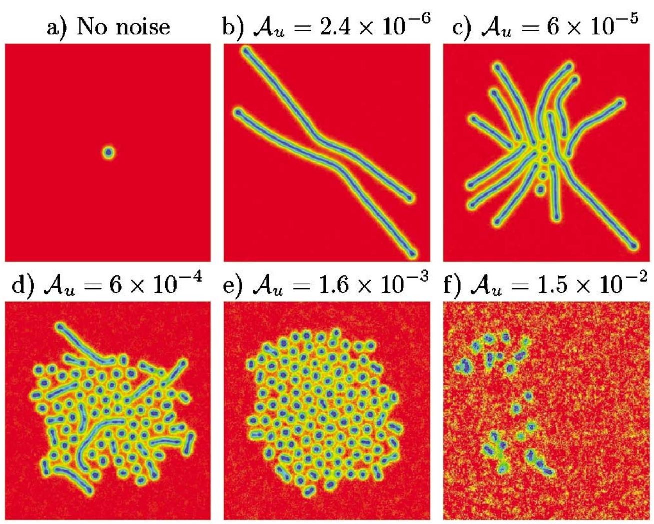

FIG. 2 (color). Noise-controlled patterns at time $t = 18\,000$.

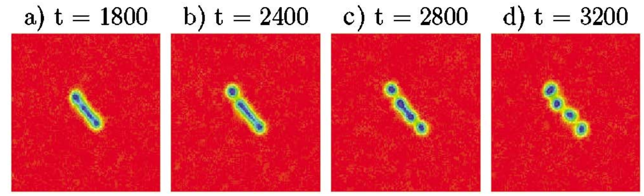

FIG. 3 (color). Breaking up of an embrionic stripe for $\mathcal{A}_u = 10^{-3}$.

diffusion from all directions, the stripe receives the substrate only perpendicularly. As a consequence, the most active areas (blue in the figures) are proportionally wider in spots than in stripes. In the same way, stripe tips are wider than the rest of the stripe. In addition to allowing wider reaction areas, the larger substrate flow may provide greater stability for the spots. This provides a clear example of noise controlled *pattern selection,* where spot morphology is dynamically favored over stripe growth for a specific range of noise amplitudes.

For still higher noise intensity, the perturbations are sufficiently strong that the spots themselves become unstable and can disappear (Fig. 4). The number of spots at a fixed time decreases as noise increases [Fig. 2(f)]. For high enough noise ($\mathcal{A}_u$ greater than 2×10^{-2}), the spots are so unstable that the system locks into the trivial (stable) state $U = 1$.

The effect of noise is manifest also in the kinematics of spot multiplication (Fig. 5). The multiplication rate of the number of spots depends not only on the rate of spot replication but also on the rate of spot degradation and separation from stripes. Propagation kinetics is not efficient for low noise intensity, for which spots coexist with stripes, as the latter compete with the former for substrate resources.

There is an optimal value of noise intensity, around $\mathcal{A}_u = 2 \times 10^{-3}$, for which spot multiplication is maximal. The existence of an optimal noise level for spot multiplication bears some similarities with the phenomena of temporal [2] and spatiotemporal [23] stochastic resonance, in which system response to an oscillating excitation is maximal for a particular level of noise. However, in this case the optimization is not related to the system response to an external stimulus but to the internal growth dynamics.

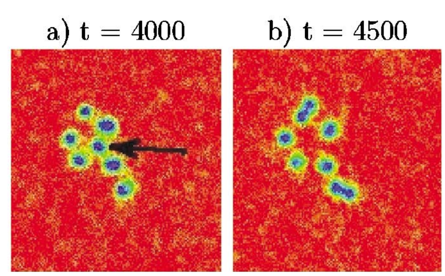

FIG. 4 (color). Spot vanishing for $\mathcal{A}_u = 6 \times 10^{-3}$. The spot indicated by the arrow in (a) disappears in (b).

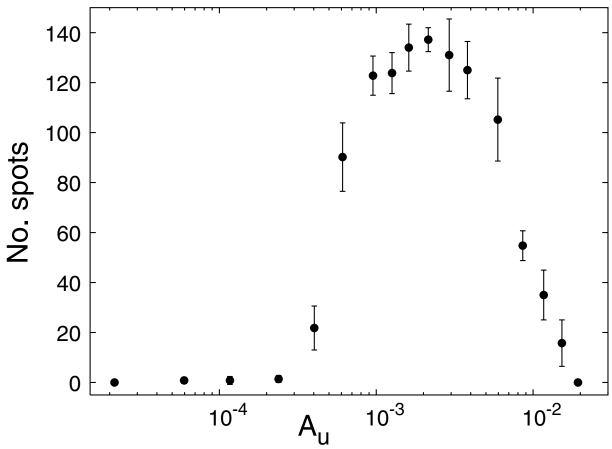

FIG. 5. Kinetics of spot multiplication. The number of spots for $t = 20\,000$ as a function of noise intensity. The error bars span 1 standard deviation.

Above the optimal level, an increase in noise intensity leads to spot instability and degradation, so that the rate of spot multiplication decreases with increasing noise.

Summary and conclusions.—To the best of our knowledge, this is the first reported case of self-organized and *self-replicating* patterns controlled by noise. Numerical simulations of a well-known stochastic reaction-diffusion model demonstrate a transition from stripe-growing elongation to spot replication driven and controlled by external noise. For the chosen parameters D, k_u, k_v we used in (1), a small amount of noise is sufficient to drive the system dynamically from one pattern, stripes, to another, namely, spots. This is achieved dynamically, induced by noise. The noise causes the model parameters to *renormalize*,, that is, to change due to the presence of the fluctuations and nonlinearities, in such a way that the renormalized effective parameters correspond to different patterns. This is supported in part by the theoretical analysis reported in [21]. There, a lowest order one-loop renormalization group calculation is performed on the stochastic GS model and the parameters ν and λ are seen to renormalize as explicit and computable functions of the random noise and the nonlinearity. The growth kinetics is also controlled by noise in such a way that there is an optimal noise intensity for which spot multiplication is maximal. For larger noise intensities, the spots become unstable and the system is attracted to the spatially homogeneous steady state. One important conclusion is that the environmental noise can select and control complex spatio-temporal patterns of potential biological and chemical relevance.

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