Propensity Criterion for Networking in an Array of Coupled Chaotic Systems

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(Received 1 July 2003; published 1 December 2003)

We examine the mutual synchronization of a one-dimensional chain of chaotic identical objects in the presence of a stimulus applied to the first site. We first describe the characteristics of the local elements, and then the process whereby a global nontrivial behavior emerges. A propensity criterion for networking is introduced, consisting in the coexistence within the attractor of a localized chaotic region, which displays high sensitivity to external stimuli, and an island of stability, which provides a reliable coupling signal to the neighbors in the chain. Based on this criterion, we compare homoclinic chaos, recently explored in lasers and conjectured to be typical of a single neuron, with Lorenz chaos.

DOI: 10.1103/PhysRevLett.91.234101 PACS numbers: 05.45.Xt, 05.40.–a, 05.50.+q

An open problem in science is how to build a semantic network on minimal assumptions. In a living brain, an external stimulus localized at some input spreads over a large assembly of coupled neurons building up a collective state univocally corresponding to the stimulus. The current conjectures and the preliminary experimental evidence [1,2] on this time dependent networking problem lead to a new paradigm wherein perceptions require mutual synchronization of neuronal spikes. A model of dynamical encoding by networks of competing neurons has been recently introduced [3]; however, the core issue of synchronizing a large number of neurons as it appears from experiments [2] has not been addressed thus far.

With reference to this issue, we introduce a minimal model built upon simplicity requirements; namely, we consider linear and symmetric interneuron coupling; furthermore, we take just nearest neighbor coupling, avoiding architecturally complicated connections. Under these assumptions we address the question of how complexity arises, calling complexity the fact that a composite system displays collective properties not directly deducible from the dynamical behavior of its constituent elements [4].

Here we focus on the collective response of an array of coupled dynamical objects to a localized external stimulus, and provide a propensity criterion for networking, that is, for organizing in a collective state univocally related to the stimulus. Having in mind time dependent networking, as is the case of biological communication, we examine mutual synchronization of a one-dimensional chain of identical objects in the presence of a stimulus applied to the first site. We first describe the characteristics of the local elements, and then the process whereby a global nontrivial behavior emerges.

Two independent requirements must be fulfilled by each element. First, it should have a sensitivity region in order to easily respond to the neighbor coupling; second, it has to provide a strong enough signal to relay the input stimulus along the whole array. The first requirement suggests to recur to dissipative chaotic systems; in fact modifying a regular individual dynamics confined to a stable attractor would imply a consistent expenditure of energy and time [5], whereas sweeping through the manifold of unstable periodic orbits which make up a chaotic attractor is a fast costless operation [6]. The second requirement is conflicting with the previous one; in fact, it implies islands of stability within the chaotic orbit, out of which to extract a reliable driver for the next neighbor.

The twofold problem is solved recurring to large spikes emerging out of a small chaotic background; indeed a weak intersite coupling will provide a discrete synchronization associated only with the large spikes, the chaotic background being not effective. This discrimination will amount to the approximation inducing the transition from an individual to a collective description. At variance with the standard chaotic synchronization scheme [7] where two identical systems synchronize along the whole orbital evolution, here the synchronization occurs because a large spike of one system is forcing the neighbor to escape away from its chaotic region, thus yielding its own spike [8,9].

Among the chaotic systems, those best suited to the emergence of a new hierarchical level should thus be characterized by temporal windows of stability and chaos within each orbit.

We give substance to these considerations with reference to the heteroclinic transfer back and forth between a saddle focus (SF) and a saddle node (SN) under the so-called Shilnikov condition [10] [see Figs. 1(a)–1(c)]. Such a behavior has been explored both experimentally [11] and theoretically [12] with reference to a CO_2 laser with feedback. For the sake of brevity, it will be called HC (homoclinic chaos) [13]. In fact, the mere homoclinic return to SF provides a chaotic transient [14] but it does not assure a regular motion away from SF; on the contrary, the further presence of SN yields the stability island necessary for networking.

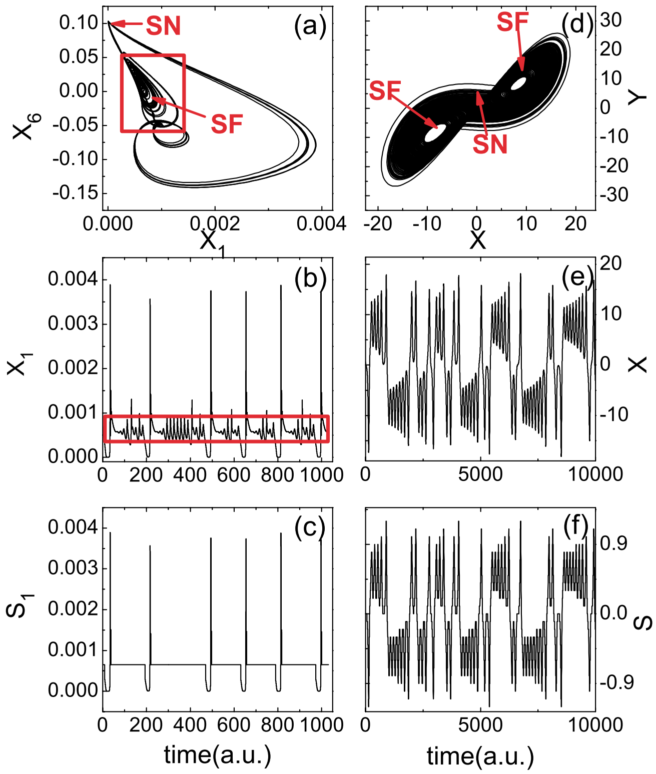

FIG. 1 (color online). Comparison between HC (homoclinic chaos) (a),(b),(c) [11] and Lorenz chaos (d),(e),(f). In (a) and (d) a phase space projection over two dynamical variables; SF denotes the saddle focus and SN the saddle node; on the left the two SF map one onto the other after an inversion ($x \to -x$, $y \to -y$) around the origin. (b) and (e) show the time series for variables x_1 of HC (it represents the laser intensity in the case of the CO_2 laser) and x of Lorenz; in the former case, a suitable threshold cuts off the chaotic background; in the latter case, no convenient region for thresholding can be isolated. In (c), after threshold, the new variable $S(t)$ alternates spikes with flat regions where the system has a high sensitivity and short refractive windows where the intensity $S(t)$ goes to zero.

The dynamics is characterized by a sequence of spikes with widely fluctuating time intervals T. Such a structure underlies spiking behavior in many neuron [15,16], chemical [17], laser [11], and El Niño [18] systems. It is important to note that this dynamics is highly nonuniform, in the sense that the sensitivity to small perturbations is high only in the vicinity of the saddle focus along the unstable directions. A weak noise thus may influence T significantly [19].

In the system we consider [12], the chaotic behavior is confined to a small neighborhood of SF where it fulfills the Shilnikov condition [10] $\gamma < \alpha$,γ being the real part of the expansion rate on the unstable manifold of SF and α the contraction rate on the stable manifold of SF. Precisely, the linearized dynamics around SF is ruled by the leading eigenvalues $(-\alpha, \gamma \pm i\omega)$ that for the considered control parameters imply $\alpha > \gamma$ [12], all the other eigenvalues having very large negative real parts. The exiting trajectories along the unstable manifold reenter the stable manifold after a large orbit in phase space, corresponding to the heteroclinic approach to SN. The phase space orbit appears as a confined chaotic tangle, concentrated around the saddle focus which is connected to a wide regular section, the two parts closing the orbit [Fig. 1(a)–1(c)].

For such a system, synchronization has been demonstrated against an external clock [13], or against a previous time slot of itself, presented after a suitable delay [20] or after a low pass filter [21]. Furthermore, HC is robust against noise [19].

We wish to stress the general aspect of HC. In the membrane dynamics of the Hodgkin-Huxley model [15] upon which the action potentials of neuronal axonsaxons are modeled, appropriate parameter ranges display HC, as discussed in Refs. [15,16]. This occurs also in many chemical reactions [17]. Even when the number of coupled variables is higher than 3, the local dynamics around the saddle focus is ruled only by three eigenvalues, the others being largely negative. Thus, we can refer to three relevant degrees of freedom, the other ones being adiabatically eliminated.

As for the FitzHugh-Nagumo simplification of Hodgkin-Huxley [22,23], it consists of two coupled variables and then it justifies the existence of only periodic or excitable regimes; in order to achieve chaos, one must increase the phase space dimensions introducing either a time varying external perturbation or some noise.

As we go from one system to an array of coupled identical systems [8,9], mutual synchronization occurs either spontaneously or as a response to an external forcing applied to a single site. We recall [9] that the coupling is realized by replacing in one of the HC equations [12] a scalar component x_1^i (i = site index) with $x_1^i + \epsilon(x_1^{i+1} + x_1^{i-1} - 2\langle x_1^i \rangle)$, where $\langle x_1^i \rangle$ denotes a moving time average. The range of coupling strengths ϵ considered here is between 0 and 0.25.

To show the advantage of HC in comparison to more conventional types of chaotic behavior, we consider the Lorenz model for the standard values of the control parameters $b = 8/3, \sigma = 10$, $r = 28$ [24], that from now on we call the standard Lorenz. In terms of fixed points, also the rectified Lorenz system (i.e., the Lorenz model plus an inversion operation around the origin) is characterized by one SF and one SN; however, the eigenvalues around SF are $(-\alpha \pm i\omega, \gamma)$ with $\alpha = 14.48$ and $\gamma = 0.4119$, thus very far away from the Shilnikov condition.

In fact, morphologically the two cases appear very different; thus, even though a homeomorphism should map one over the other, introduction of extra operations, such as a sensitivity threshold, renders such a transfer impossible, as one can see by comparing before [Fig. 1(e)] and after thresholding [Fig. 1(f)].

Precisely, the time plot of one of the HC variables consists of a train of identical spikes, separated by a variable interspike interval (ISI) filled with a rather small chaotic background confined within a stripe thinner than 1/5 of the spike height.

As we couple a large number of such dynamical system in a network, for a sufficiently weak coupling the chaotic background of Fig. 1(b) will not induce a relevant perturbation as instead it occurs for the large spikes; thus, for this purpose the coupling signal appears as in Fig. 1(c). Later, below, we discuss a quantitative limit to the intersite coupling strength which implies a natural thresholding, that is, the natural one of the chaotic amplitude background.

The emerging spikes display a chaotic time occurrence, which is epitomized by the correlation properties of ISIs. Mathematically, this thresholded chaos is described as $S(t) = S_o \sum_l \delta(t - \tau_l)$, where τ_l are the chaotic spike positions and $(\mathrm{ISI})_l = \tau_l - \tau_{l-1}$. In he presence of an external driving signal, the spikes can synchronize to it [13]; in the case of many coupled systems of this type located on an array, wide parameter ranges are found within which the individual sites mutually synchronize their own spikes, so that the space-time plot of the spike positions at each site appears as a regular fabric [8,9].

Such an easy mutual synchronization as the response to a localized input represents a semantic property absent in the Lorenz case [7], where there is no apparent scale separation where to consider a threshold. We thus identify the propensity criterion with the presence within each orbit of two very different amplitude scales—that of the large spikes and that of the small chaotic background—the mutual coupling between sites being operated by the spikes whereas the background represents the high sensitivity region within which the coupling takes place.

This is by no means limited to the discussed example, but it is generic of any dynamical trajectory passing through a saddle focus under the Shilnikov condition, including those situations which characterize biological oscillations [15,16] or chemical instabilities [17].

As an indicator of successful networking, we take the coherence parameter [25]

$$R_i = \frac{\langle \mathrm{ISI}_i \rangle}{\delta \mathrm{ISI}_i}, \tag{1}$$

where $\delta\mathrm{ISI}_i$ is the square root of the ISI_i variance $[\delta\mathrm{ISI}_i = \sqrt{\langle (\mathrm{ISI}_i - \langle \mathrm{ISI}_i \rangle)^2 \rangle}]$. R_i has been used to measure the amount of synchronization of a single system to a periodic stimulus; when synchronization propagates along a chain, R can be measured anywhere since it has almost the same value on all sites. The stimulus consists of a sinusoidal perturbation of frequency ω applied to a control parameter in the first site of the chain, with an amplitude of 30% of the unperturbed parameter value. Precisely, the bias B in the feedback amplifier is modulated as $B(t) = B_0[1 + A\sin(\omega t)]$, with $A = 0.3$ [9].

Notice that standard chaotic synchronization can propagate along a chain of generic chaotic systems; however, in general there is no propensity of the first site to synchronize to a weak external input, as shown by the corresponding indicator R (Fig. 2) for the systems of Fig. 1, either in the case of propensity [Fig. 1(a)–1(c)] or no propensity [Fig. 1(d)–1(f)].

Thus, it is one thing to mutually synchronize many identical chaotic objects, as occurs both for HC and Lorenz, but it is a different thing to synchronize the array to an external stimulus, which is feasible for HC and unfeasible for Lorenz as shown in Fig. 2.

There it can be seen that, for a single site, the R value is about 30 for ω at the natural frequency of the system $\omega_0 = 2\pi/\langle \mathrm{ISI} \rangle$, and increases up to 10^4 for larger frequencies, whereas for the Lorenz case it is consistently $R = 1$ (no coherence).

This fact proves the lack of propensity of the Lorenz system as compared to the HC where the coherence R increases by 4 orders of magnitude.

We now address the crucial question of how the coherence R_1 induced on the first site $i = 1$ propagates along the array, for different coupling strengths ϵ and frequencies ω of the input signal. As shown in Fig. 3, reducing the coupling from $\epsilon = 0.25$ to $\epsilon = 0.08$ reduces R by 3 orders of magnitude, thus showing the natural thresholding effect occurring in a network of HC systems, without having to explicitly take care for the operation leading from Fig. 1(b) to Fig. 1(c).

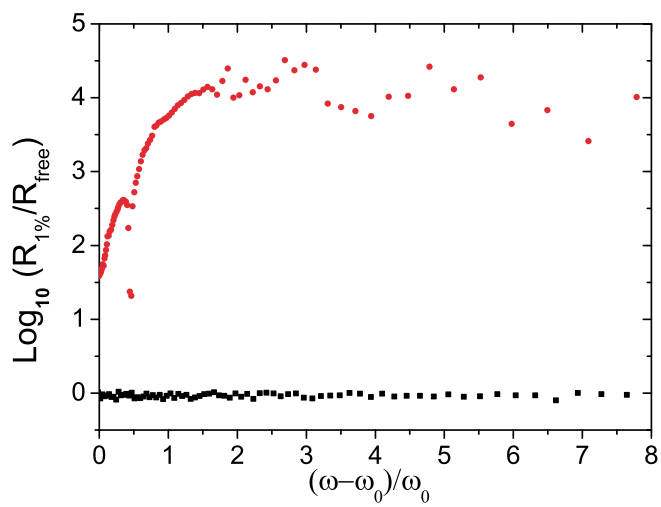

FIG. 2 (color online). Coherence parameter $R_{1\%}$ for a driving signal consisting of a 1% periodic perturbation of a control parameter normalized to R_{free} (the R value in the absence of perturbation), plotted versus the normalized distance of the perturbation frequency from the natural value ω_0. Circles: HC; squares: Lorenz. For HC the ratio $R_{1\%}/R_{\mathrm{free}}$ is about 30 at ω_0 and it increases up to 10^4 for ω, twice or more the natural frequency; indeed, since synchronization means forcing HC away from the SF region, frequencies higher than the natural one are easier to synchronize, whereas for smaller frequencies HC may have a spontaneous escape from SF. To smooth the plot, a small amount of Gaussian white noise (0.5% rms) has been added to the bias parameter. For the Lorenz case, the indicator is always at 1.

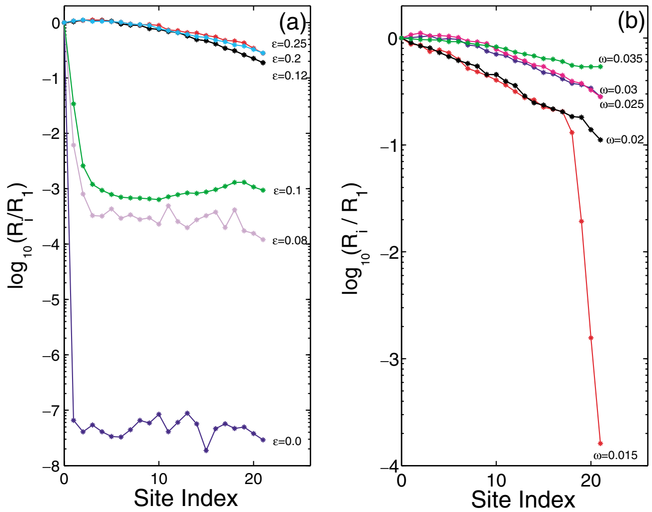

FIG. 3 (color online). Propagation along a chain of $N = 20$ sites of the coherence value R_1 of the first site exposed to an external stimulus: (a) for a fixed input frequency close to the natural one ω_0 and different coupling strengths ϵ; (b) for $\epsilon = 0.2$ and different external frequencies ω, with $\omega_0 = 0.02$ the natural frequency.

Notice that the same property had been measured in Ref. [9]; here, however, we provide a quantitative assignment of the coupling strength necessary in order to cut off the chaotic background and thus induce the equivalence of Figs. 1(b) and 1(c) without the need for filtering operations.

In conclusion, we have introduced the notion of a semantic network as an array of coupled identical chaotic systems which assume a collective state in the presence of a localized periodic stimulus. The propensity criterion appears morphologically as a confinement of the chaotic tangle within a small region of the total attractor; the corresponding indicator is a tremendous increase in the R parameter near the natural frequency.

At variance with complete synchronization [7], here we consider only synchronization of large spikes, intercalated by intervals not effective in acting over the neighbors, yet highly sensitive to external signals. This split of the dynamical orbit into two different regions is the condition to build a nontrivial collective state as a response to a localized stimulus.

The authors are indebted to S. Boccaletti and P. Lauro-Grotto for useful discussions. E. A. and I. L. acknowledge support from the European Contract CoSyC of SENS No. HPRN-CT-2000-00158.

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