Minimal Brownian Ratchet: An Exactly Solvable Model

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We develop an analytically solvable three-state discrete-time minimal Brownian ratchet (MBR), where the transition probabilities between states are asymmetric. By solving the master equations, we obtain the steady-state probabilities. Generally, the steady-state solution does not display detailed balance, giving rise to an induced directional motion in the MBR. For a reduced two-dimensional parameter space, we find the null curve on which the net current vanishes and detailed balance holds. A system on this curve is said to be balanced. On the null curve, an additional source of external random noise is introduced to show that a directional motion can be induced under the zero overall driving force.

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The Brownian ratchet and pawl system was first correctly explained by Smoluchowski [1] and later revisited by Feynman [2]—this has inspired much activity in the area of Brownian ratchets, despite flaws in Feynman's analysis of the thermal efficiency of the ratchet engine [3] and detailed balance [4].

Interest has revived because molecular motors [5] have been described in terms of Brownian ratchet [6,7] models. Another area of interest has been in Parrondo's paradox [8], where losing strategies cooperate to win. This can be illustrated in terms of games that lose when played individually, but win when alternated—this has been shown to be a discrete-time Brownian ratchet [9], otherwise known as a "Parrondian game." Parrondo's games have significantly sparked recent interest in the areas of lattice gas automata [10], spin models [11], random walks and diffusions [12–14], biogenesis [15], molecular transport [16,17], noise induced patterns [18], stochastic control [19,20], stochastic resonance [21], and quantum game theory [22,23]. Recently, Reimann [24] has performed an extensive review of the ratchet field.

Jarzynski *et al.* [25] developed an exactly solvable Brownian ratchet that can be operated as a heating system or refrigerator, depending on the parameters between two heat reservoirs of different temperatures. However, this is treated as a six-state system and solution is via matrix inversion of coupled linear equations. The derivation is somewhat complex, so the physical picture and key ingredients of the observed properties are obscured.

Westerhoff *et al.* [26] have analyzed enzyme transport using a four-state model. In this paper, for the first time, we develop a three-state discrete-time Brownian ratchet model that can be solved analytically. We call it the *minimal Brownian ratchet* (MBR) [27]. By setting up and solving the steady-state solution of the corresponding master equations, we obtain the probability current and the null surface, in the parameter space, of the noisy and noise-free MBR. The obtained solution does not show any critical behavior and can be suitably explained in terms of nonsingular behaviors.

The minimal ingredients of a Brownian ratchet are an asymmetric potential and random noise. In Fig. 1, we show the state diagram of the MBR. The MBR has three states, $\{S_0, S_1, S_2\}$, where the transition probabilities between states are asymmetric. The transition probability that a random walker in state S_k steps in the positive

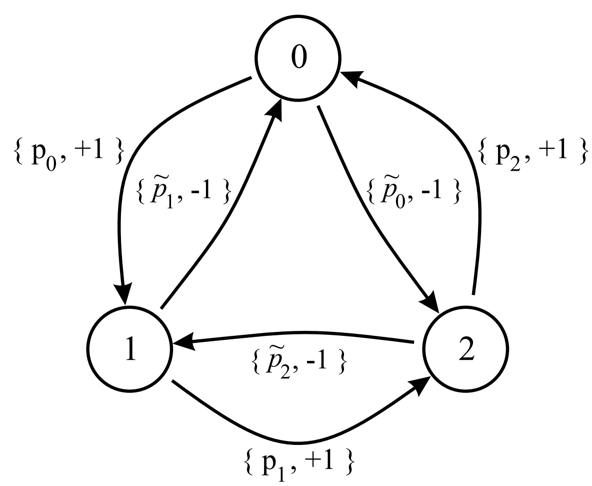

FIG. 1. State-transition diagram of a three-state discrete-time Brownian ratchet with asymmetric transition probabilities p_0, p_1, and p_2 in the positive direction (counterclockwise) and $(1 - p_0)$, $(1 - p_1)$, and $(1 - p_2)$ in the negative direction (clockwise). Each transition has two numbers associated with it; $\{p_k, R_k\}$. The first number in the brackets, p_k, is the conditional probability of that transition (given the initial state). The second number, R_k, is the reward associated with that transition. Note that we have a skip-free process, which means the reward structure is $+1$ for "winning" transitions and -1 for "losing" transitions.

direction is p_k. The probability of a shift in the negative direction is $\tilde{p}_k$. This is true for $k \in \{0, 1, 2\}$. We define the positive direction as counterclockwise. The condition of normalization, $p_k + \tilde{p}_k = 1$, is automatically enforced by our choice of symbols. These ingredients comprise a three-state random walk model with generalized asymmetric potential and we call it a noise-free MBR.

It is straightforward to set up the following difference equations for the probability distributions of the noise-free MBR model:

$$P_k(t+1) = P_{k+1}(t)\tilde{p}_{k+1} + P_{k+2}(t)p_{k+2}, \quad (1)$$

for all cyclic (modulo-3) state indices k. $P_k(t)$ is the probability for the random walker at time t to be on the state of S_k. This can be written in matrix form as $\mathbf{P}_{t+1} = \mathbf{P}_t B$, where $\mathbf{P}_t$ is the time varying probability (row) vector at time t and B is the transition probability matrix. We can write

$$[B_{i,j}] = \begin{bmatrix} 0 & p_0 & \tilde{p}_0 \\ \tilde{p}_1 & 0 & p_1 \\ p_2 & \tilde{p}_2 & 0 \end{bmatrix}.$$

The steady-state probability, after a sufficiently long time, $\lim_{t\to\infty} \mathbf{P}_t = \mathbf{P}$ is simply given as

$$\mathbf{P} = \mathbf{P}B, \quad (2)$$

which is a characteristic value problem. A partial probability current, I, can be defined as

$$I = P_k p_k - P_{k+1}\tilde{p}_{k+1}. \quad (3)$$

If $I = 0$, there is no net current and detailed balance [28] is satisfied, otherwise there exists a net current and the system will assume a nonequilibrium steady state.

Solving Eq. (2) together with the normalization condition, $P_0 + P_1 + P_2 = 1$, is again straightforward. Using the standard methods for characteristic value or eigenvalue problems, we obtain

$$P_k = (\tilde{p}_{k+1} + p_{k+1}p_{k+2})/D, \quad (4)$$

for all k. The denominator D is given as

$$D = 2 + p_0 p_1 p_2 + \tilde{p}_0\tilde{p}_1\tilde{p}_2. \quad (5)$$

These expressions are consistent with the results of Pearce [29]. It is easy to check that they are the solution to Eq. (2) by direct substitution.

We can substitute the results from Eq. (4) into Eq. (3) to solve for the net current, I:

$$I = (p_0 p_1 p_2 - \tilde{p}_0\tilde{p}_1\tilde{p}_2)/D. \quad (6)$$

The condition for detailed balance $I = 0$ is then

$$p_0 p_1 p_2 = \tilde{p}_0\tilde{p}_1\tilde{p}_2, \quad (7)$$

which is the equation of a two-dimensional surface in the three-dimensional parameter space, $\{p_0, p_1, p_2\}$. Note that Eq. (6) is independent of state index k as is required by its definition given in Eq. (3).

The second part of the MBR is to introduce additional random noise to the system. To the noise-free MBR, we add more noise, controlled by the parameter γ, to the MBR as follows. With a probability of $\tilde{\gamma}$, a random walker follows the dynamic rule of the noise-free MBR otherwise, with the probability of γ $(=1-\tilde{\gamma})$, the walker randomly takes a right or left step with the equal probability of a half. For $\gamma = 0$, the model is exactly the same with the noise-free MBR. In the other limit, for $\gamma = 1$, the randomizing process dominates and the system reduces to a simple unbiased random walk where the net current remains zero. It is important to note that γ *influences* the level of noise in the ratchet but is not *identical* with the noise itself. We refer to γ as a "noise parameter." With this modification the transition probability matrix B changes as

$$[B_{i,j}] = \begin{bmatrix} 0 & \tilde{\gamma}p_0 + \gamma/2 & \tilde{\gamma}\tilde{p}_0 + \gamma/2 \\ \tilde{\gamma}\tilde{p}_1 + \gamma/2 & 0 & \tilde{\gamma}p_1 + \gamma/2 \\ \tilde{\gamma}p_2 + \gamma/2 & \tilde{\gamma}\tilde{p}_2 + \gamma/2 & 0 \end{bmatrix}.$$

From the transition matrix, we know that adding the random noise with parameter γ effectively changes the existing parameters as

$$p_k \to \tilde{\gamma}p_k + \gamma/2, \quad (8)$$

and the same holds for $\tilde{p}_k$. The steady-state solution and net current for noisy MBR can be obtained by exchanging all p_k in Eqs. (4) and (6) according to Eq. (8). The expression for current is given as

$$I_\gamma = [\tilde{\gamma}^3 A_- + \tilde{\gamma}(\gamma/2)(1-\gamma/2)B]/D_\gamma, \quad (9)$$

where

$$\begin{aligned} D_\gamma &= 2 + \tilde{\gamma}^2 A_+ + \tilde{\gamma}(\gamma/2)(1+\gamma/2), \\ A_\pm &= p_0 p_1 p_2 \pm \tilde{p}_0\tilde{p}_1\tilde{p}_2, \quad (10) \\ B &= p_0 + p_1 + p_2 - \tilde{p}_0 - \tilde{p}_1 - \tilde{p}_2. \end{aligned}$$

It is possible to further restrict the choices of $\{p_0, p_1, p_2\}$ without losing the important properties of the ratchet. Parrondo's original definition imposed the further constraints $p_0 = q$ and $p_1 = p_2 = p$. This reduced the parameter space to a two-dimensional space with parameters $\{p, q\}$. In two-dimensional $\{p, q\}$ parameter space, the condition of detailed balance, i.e., Eq. (7), gives the equation for a curve that we call the null curve:

$$q = \frac{\tilde{p}^2}{p^2 + \tilde{p}^2} = \frac{(1-p)^2}{p^2 + (1-p)^2}. \quad (11)$$

The null curve is a special case of the more general null surface or null hypersurface, in higher dimensions. Figure 2 shows the "positive" and "negative" net current

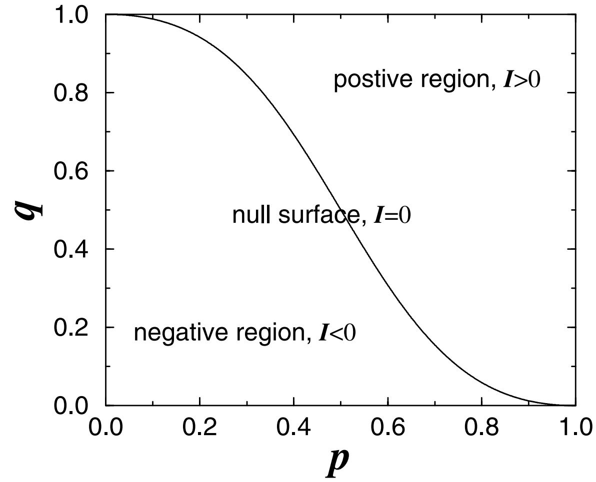

FIG. 2. The null surface of a three-state discrete-time Brownian ratchet. On the null surface, $q = (1-p)^2/[p^2 + (1-p)^2]$, the current vanishes. Above the curve, the system has a positive net current. Below the curve, the system has a negative net current.

regions of the noise-free MBR. Note that as expected from the symmetry of the system the curve is invariant under the transformations $q \rightarrow (1-q)$ and $p \rightarrow (1-p)$. This is also apparent from a consideration of Eq. (7).

On the null surface, we introduce additional random noise to the system by controlling the value of γ. For $\gamma = 0$, the model is exactly the same as the noise-free MBR, and the net current remains zero since we are on the null surface. In the other limit, for $\gamma = 1$, the randomizing process dominates and the system reduces to a simple unbiased random walk where the net current is also zero. However, counterintuitively, for $0 < \gamma < 1$ nonzero current is induced by introducing random noise controlled by γ.

In Fig. 3, we show the current versus noise parameter, γ, for different values of parameters p and $q = \tilde{p}^2/(p^2 + \tilde{p}^2)$. As γ is increased from zero, the current increases to a maximum and then falls off, which has the form of stochastic resonance [21]. The position of this extremum can be obtained from the condition, $\partial I_\gamma/\partial\gamma = 0$. γ varies a little from 0.408 for $p = 0.1$ to 0.423 for $p = 0.5$.

Figure 4 shows the net current, I_γ, versus noise γ when the system is not on the null surface any more. In the off-balance region, $q \neq \tilde{p}^2/(p^2 + \tilde{p}^2)$, the net current is not zero for $\gamma = 0$ but still should be zero for $\gamma = 1$ and the intermediate behavior is qualitatively the same as the balanced behavior. The actual values of p and q for the various curves in Fig. 4 are in linear increments of 0.01 for p and q. The top curve has parameters $p = 0.77$ and $q = 0.12$. The bottom curve has parameters $p = 0.73$ and $q = 0.08$.

We can generalize the MBR by introducing a bias into the added noise. The walker takes a right step with

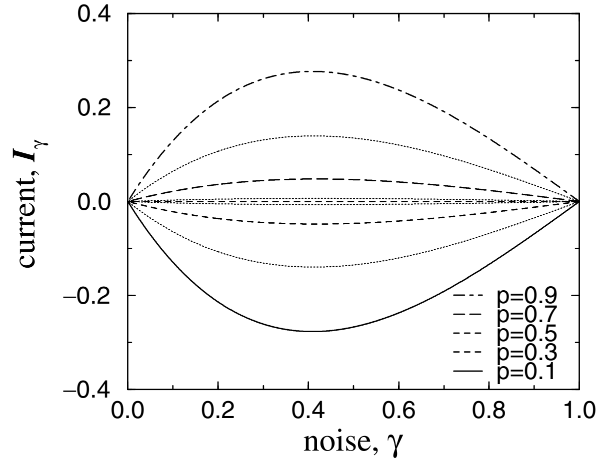

FIG. 3. The probability current, I_γ, versus noise parameter, γ, on the null surface. For values of $p \neq 0.5$, additional noise induces a net current that increases, in magnitude, with increasing γ and then decreases, in magnitude, to zero after γ exceeds an optimum value. The bottom curve corresponds to $p = 0.1$. All the other curves represent increments of $\Delta p = 0.1$. The middle curve corresponds to $p = 0.5$. The top curve corresponds to $p = 0.9$. Parrondo's original games had $p = 0.75$.

probability of $0.5 - \epsilon$ and a left step with probability of $0.5 + \epsilon$. For $\epsilon \neq 0$ this noise introduces nonzero net current. The new parameter, ϵ, is essentially a measure of the bias in the added noise. This generalized model qualitatively shows the same behavior as the noisy MBR.

We can generalize this model to a system of size N by repeating the unit cell of modulo-3 N times with a periodic boundary condition. In this case, the periodic potential ensures $p_k(t) = p_{k+3n}(t)\ \forall\ n = 0, \pm1, \pm2, \ldots$. Because of the normalization condition, $\sum_{k=1}^{N} P_k(t) = 1$, the current will be reduced by a factor of N. Otherwise,

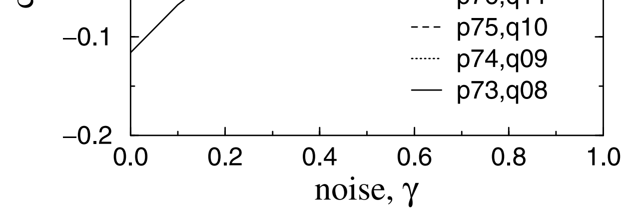

FIG. 4. The probability current I_γ versus noise parameter γ off the null surface. The bottom curve corresponds to $p = 0.73$ and $q = 0.08$ and the top to $p = 0.77$ and $q = 0.12$.

the corresponding master equations and solutions are exactly the same as the minimal model.

For different moduli, in principle, we can also set up the master equations and solve them exactly by matrix inversion for the set of linear equations. It can be shown that these results have qualitatively the same statistical behavior as the three-state MBR. Note that for even number moduli there are oscillatory nonstationary solutions.

The transformation in Eq. (8) tells us effectively that the MBR gives the same results as a biased random walk, where the transition probability is not symmetric but biased. Although this analogy can be used to investigate the characteristics of MBR, it is absolutely impossible to determine whether the system is itself a biased random walk or an MBR by analyzing the result of measurements, without prior knowledge that the model is a combination of a balanced unbiased random walk and added random external noise. This makes the MBR valuable for understanding the minimal features of the discrete-time Brownian ratchet. The MBR has applicability in discrete-time processes where the transition probabilities do not fluctuate in time, such as in game and computation theory where transitions occur at precisely defined times.

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