Quantum States far from the Energy Eigenstates of Any Local Hamiltonian

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What quantum states are possible energy eigenstates of a many-body Hamiltonian? Suppose the Hamiltonian is *nontrivial*, i.e., not a multiple of the identity, and L *local*, in the sense of containing interaction terms involving at most L bodies, for some fixed L. We construct quantum states ψ which are "far away" from all the eigenstates $\mathbf{E}$ of any nontrivial L-local Hamiltonian, in the sense that $\|\psi - \mathbf{E}\|$ is greater than some constant lower bound, independent of the form of the Hamiltonian.

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A central problem in physics is the characterization of eigenstates of many-body Hamiltonians. Less attention has been devoted to the complementary question: which quantum states are not the eigenstates of any physically plausible Hamiltonian? The purpose of this Letter is to address this question, by explicitly constructing states which are, in a sense made precise below, far away from the eigenstates of any nontrivial, local Hamiltonian. Such constructions are interesting for several reasons. First, they place fundamental restrictions on the physics of many-body quantum systems. Second, as we discuss in detail below, our construction gives insights into the construction of "naturally fault-tolerant" quantum systems that are able to resist the effects of noise and decoherence.

The Letter begins with a simple counting argument showing that "most" quantum states are not the eigenstates of any physical Hamiltonian. We then give a more powerful—albeit, still quite simple—argument constructing quantum states ψ far away from all the eigenstates $\mathbf{E}$ of any nontrivial, L-local Hamiltonian. In this statement, by nontrivial we mean not a multiple of the identity [1], and by L local we mean that each interaction term in the Hamiltonian involves at most L bodies. Of course, physically we expect that L is a small constant, 2, or at most 3 in special circumstances. Quantitatively, for an n-body system whose constituents have d-dimensional state spaces, we prove $\|\psi - \mathbf{E}\| \geq [(L+1)\binom{n}{L}(d^2-1)^L]^{-1/2}$. What is interesting about this bound is that it is a *constant* lower bound that holds for the eigenstates of *all* nontrivial L-local Hamiltonians, even those with degenerate eigenstates.

It is worth noting that the results reported in this Letter hold unchanged for any n-local *observable,* not just Hamiltonians. However, particularly in the light of recent work characterizing the entangled properties of the ground states of lattice systems [2], the case of the Hamiltonian is of especial interest, and we prefer this nomenclature throughout.

Interestingly, the states ψ we construct are special examples of *quantum error-correcting codes* [7]; such codes turn out to be rich sources of states which are not close to being eigenstates of any nontrivial, local Hamiltonian. Our Letter thus illustrates a general idea discussed elsewhere [10–13], namely, that quantum information science may provide useful tools and perspectives for understanding the properties of complex quantum systems, complementary to the existing tools used in quantum many-body physics.

We begin with a counting argument showing most quantum states cannot arise as energy eigenstates of local Hamiltonians. This counting argument has the advantage of simplicity, but also has some significant deficiencies, discussed and remedied below. Suppose an n-body quantum system is described by an L-local Hamiltonian, H. We suppose, for simplicity, that each quantum system has a two-dimensional state space; that is, the systems are "qubits," in the language of quantum information science. It is straightforward to adapt the argument below when the component systems have state spaces with higher dimensionalities, and also when different systems have different dimensionalities.

It will be convenient to expand our Hamiltonian as

$$H = \sum_\sigma h_\sigma \sigma, \tag{1}$$

where h_σ are real coefficients, and the σ denote tensor products of the Pauli matrices $I, \sigma_x, \sigma_y, \sigma_z$. For an L-local Hamiltonian, we see that $h_\sigma = 0$ whenever the *weight* of σ—that is, the number of nonidentity terms in the tensor product—is greater than L.

The number of independent real parameters h_σ[14] occurring in Eq. (1) is

$$\#(n, L) = \sum_{j=0}^{L} \binom{n}{j} 3^j. \tag{2}$$

To see this, note that the different terms in the sum come from the interactions involving $j = 0, 1, \ldots, L$ bodies, respectively. For the j-body interactions, there are $\binom{n}{j}$ ways of picking out a subset of j interacting systems, and given a particular subset the number of parameters is 3^j, corresponding to the 3^j nontrivial tensor products of Pauli operators. When $L \leq n/2$ we obtain a useful upper bound on $\#(n, L)$ by noting that $\binom{n}{j} \leq \binom{n}{L}$, and $3^j \leq 3^L$:

$$\#(n, L) \leq (L + 1)\binom{n}{L}3^L. \tag{3}$$

For real physical systems we expect $L = 2$ or (rarely) $L = 3$, for which

$$\#(n, 2) = \frac{9n^2 - 3n + 2}{2}; \tag{4}$$

$$\#(n, 3) = \frac{9n^3 - 18n^2 + 15n + 2}{2}. \tag{5}$$

More generally, for any fixed L, $\#(n, L)$ is a polynomial of degree L in n.

Next, consider the set of states which can be obtained as the nondegenerate ground state [15] of an L-local Hamiltonian. This set can be parametrized by $\#(n, L)$ real parameters. Since an arbitrary state of n qubits requires $2 \times 2^n - 2$ real parameters to specify, provided $\#(n, L) < 2 \times 2^n - 2$, we see that there exists a state ψ which cannot arise as the nondegenerate ground state of any L-local Hamiltonian. Comparing with the bound Eq. (3) we see that this is generically the case except in the case where L approaches n; that is, unless, the number of bodies interacting approaches the number of bodies in the system. For large values of n this is an unphysical situation, and generic quantum states will *not* be the ground state of a nondegenerate L-local Hamiltonian.

This argument proves the existence of quantum states which are not eigenstates of any nondegenerate, L-body Hamiltonian. However, there are many deficiencies with the argument. First, the argument establishes only the *existence* of such states; it does not tell us what they are. Second, while the argument shows that such a state cannot be an *exact* eigenstate, it does not provide any limitation on how close it can be to an eigenstate. Indeed, phenomena such as space-filling curves show that a manifold of a small dimension can "fill up" a manifold of a larger dimension so that every point in the manifold of the larger dimension is arbitrarily close to a point in the manifold of the smaller dimension. Third, the argument requires the eigenstates to be nondegenerate. This deficiency may be partially remedied by noting that the manifold of states arising as eigenstates of Hamiltonians with up to m-fold degeneracy is at most $m \times \#(n, L)$ dimensional. However, as m increases, the bound obtained by parameter counting becomes weaker and weaker.

A much stronger argument can be obtained using the theory of quantum error-correcting codes (QECCs). We now briefly introduce the relevant elements of the theory of QECCs and explain a simple observation motivating the connection between L-local Hamiltonians and QECC states. Then, below, we develop a stronger quantitative version of the argument.

The idea of quantum error correction is to encode the state of a small physical system, such as a qubit, in a larger quantum system, such as a collection of qubits. The hope is that the encoded state will be more robust against noise than if it were not encoded. This hope was realized in schemes proposed by Shor [16] and Steane [17], and since developed extensively elsewhere [18].

For example, a code encoding k qubits into n qubits is a 2^k-dimensional subspace of the 2^n-dimensional state space of n qubits. It is convenient to give the code space a label, V. We say that the code can correct errors on up to t qubits if the subspaces σV are all orthogonal to one another, for σ of weight up to t. The idea is that the different σ correspond to different error processes that may occur on the qubits. Because the σV are orthogonal it is possible to perform a measurement to determine which error occurred, and then return the system to its original state. Of course, this does not address what happens when errors occur that are not simply products of Pauli matrices on t qubits; perhaps some small random phase rotation occurs. Remarkably, it turns out that quantum error correction also works for errors which are not products of Pauli matrices; see Chap. 10 of [8] for details.

Strictly speaking, we have described a special type of quantum error-correcting code, and it is possible to find codes not of this type. In particular, for a class of codes known as *degenerate codes,* different errors σ and σ' may have *identical* effects on the code space, so σV and $\sigma' V$ are not orthonormal. However, for our purposes the nondegenerate codes we have described above are sufficient. In particular, there are many useful bounds on the existence of nondegenerate codes. We now describe an example of such a bound. The bound is the quantum Gilbert-Varshamov bound, which shows that a code of this type encoding k qubits into n qubits, and correcting errors on up to t qubits, exists whenever [19]

$$\#(n, 2t) < \frac{2^{2n} - 1}{2^{n+k} - 1}. \tag{6}$$

In the limit of large n this becomes [24]

$$\frac{k}{n} < 1 - H\left(\frac{2t}{n}\right) - \frac{2t}{n}\log(3), \tag{7}$$

where $H(x) \equiv -x\log(x) - (1 - x)\log(1 - x)$ is the binary entropy, and all logarithms are taken to base 2.

The Gilbert-Varshamov bound applies even when $k = 0$. Thus there exists a one-dimensional quantum code—that is, a quantum state, ψ—such that the states $\sigma\psi$ are all orthogonal to one another. This is true for σ up to weight t for any t satisfying

$$\#(n, 2t) < \frac{2^{2n} - 1}{2^n - 1}. \tag{8}$$

In the large n limit, this becomes $t/n < 0.0946$. Summarizing, the quantum Gilbert-Varshamov bound tells us that there exists a quantum state ψ such that the states $\sigma\psi$ form an orthonormal set for σ of weight at most t, for any t satisfying $\#(n, t) < (2^{2n} - 1)/(2^n - 1)$.

Let us return to the problem of Hamiltonians and eigenstates. Suppose ψ is a state such that $\sigma\psi$ form an orthonormal set for σ of weight at most t; ψ might be a QECC state, as above. Expanding H in the form of Eq. (1), we see that, provided $L \leq t$, $H\psi$ contains terms orthogonal to ψ unless $h_\sigma = 0$ for all $\sigma \neq I$. Thus, unless H is completely degenerate, ψ cannot be an eigenstate of H. This suggests that QECC states are interesting examples of states that cannot be eigenstates of local Hamiltonians. This is somewhat surprising in light of the fact that QECC states can be prepared efficiently, i.e., in time polynomial in n, on a quantum computer [21]. Indeed, the argument addresses two of the problems with the parameter counting argument, namely, finding a constructive procedure to find the desired states, ψ, which can be done using the methods of quantum error-correction [25], and dealing with degeneracies in H. However, it leaves the most significant problem open, namely, proving bounds on how close ψ can be to an eigenstate of H. Remarkably, the answer turns out to be "not very," as we now prove.

Suppose an n-body L-local quantum system is described by a nontrivial Hamiltonian H. We suppose H acts on qubits; the extension to other systems is straightforward. Suppose $\mathbf{E}$ is any energy eigenstate for the system, with corresponding energy E, and let $H' \equiv H - EI$ be a rescaled Hamiltonian such that $\mathbf{E}$ has energy 0. Note that $H' = \sum_\sigma h'_\sigma \sigma$, where $h'_I = h_I - E$, and $h'_\sigma = h_\sigma$ for all other σ. Let ψ be a state such that $\sigma\psi$ forms an orthonormal set for σ of weight up to L, such as a QECC state correcting errors on $t \geq L$ qubits. Introducing the operator norm $\|A\| \equiv \max_{\phi:\|\phi\|=1} \|A\phi\|$, we have

$$\|H'(\psi - \mathbf{E})\| \leq \|H'\| \|\psi - \mathbf{E}\|. \tag{9}$$

Substituting $H'\mathbf{E} = 0$, we obtain

$$\|\psi - \mathbf{E}\| \geq \frac{\|H'\psi\|}{\|H'\|}. \tag{10}$$

We can assume $\|H'\| \neq 0$, since we have assumed that H is nontrivial; i.e., it is not a scalar multiple of the identity. Now, since the states $\sigma\psi$ are orthonormal for all σ with weight at most L, we see that

$$\|H'\psi\| = \sqrt{\sum_\sigma h'^2_\sigma} = \|h'\|_2, \tag{11}$$

where $\|\cdot\|_2$ is the Euclidean, or l_2, norm for a vector. Furthermore, by the triangle inequality for norms,

$$\|H'\| \leq \sum_\sigma |h'_\sigma| \|\sigma\| = \sum_\sigma |h'_\sigma| = \|h'\|_1, \tag{12}$$

where $\|\cdot\|_1$ denotes the l_1 norm of a vector, i.e., the sum of the absolute value of the components. Substituting Eqs. (11) and (12) into Eq. (10), we obtain

$$\|\psi - \mathbf{E}\| \geq \frac{\|h'\|_2}{\|h'\|_1}. \tag{13}$$

The Cauchy-Schwartz inequality tells us that $\|h'\|_1 \leq \sqrt{\#(n, L)}\|h'\|_2$, where $\#(n, L)$ is the dimension of the vector h'. Thus we have the general bound

$$\|\psi - \mathbf{E}\| \geq \frac{1}{\sqrt{\#(n, L)}}. \tag{14}$$

Equation (14) provides a constant lower bound on the distance of ψ from any energy eigenstate $\mathbf{E}$ of H, completely independent of any details about H, other than the fact that it is a nontrivial, L-local Hamiltonian, acting on n qubits.

A stronger bound than Eq. (14) can be obtained from Eq. (13). To obtain such a bound we use calculus to remove the dependence of the right-hand side of Eq. (13) on the (unknown) parameter E, giving

$$\frac{\|h'\|_2}{\|h'\|_1} \geq \frac{1}{\sqrt{1 + \frac{(\sum_{\sigma \neq I} |h_\sigma|)^2}{\sum_{\sigma \neq I} h_\sigma^2}}}, \tag{15}$$

and thus

$$\|\psi - \mathbf{E}\| \geq \frac{1}{\sqrt{1 + \frac{(\sum_{\sigma \neq I} |h_\sigma|)^2}{\sum_{\sigma \neq I} h_\sigma^2}}}. \tag{16}$$

Note that Eq. (14) can be recovered from Eq. (16), using a Cauchy-Schwartz argument similar to that above.

These results, Eqs. (14) and (16), carry over directly to *qudit* systems (i.e., d-dimensional quantum systems), provided the operator basis σ we expand in is unitary. The only differences are that (a) the coefficients h_σ in Eq. (16) may be complex, and thus it is necessary to work with their modulus, rather than their actual value; and (b) the value of $\#(n, L)$ in Eq. (14) is somewhat larger for qudit systems. Combining these results also with Eq. (3), we may summarize these results as a theorem.

Theorem. Let H be a nontrivial L-local Hamiltonian acting on n qudits. Let ψ be a state such that the states $\sigma\psi$ are orthonormal for all σ of weight up to L. (For example, ψ might be a QECC correcting errors on up to L qubits.) Then the following chain of inequalities holds:

$$\|\psi - \mathbf{E}\| \geq \frac{1}{\sqrt{1 + \frac{(\sum_{\sigma \neq I} |h_\sigma|)^2}{\sum_{\sigma \neq I} |h_\sigma|^2}}} \tag{17}$$

$$\|\psi - \mathbf{E}\| \geq \frac{1}{\sqrt{\#(n, L)}} \tag{18}$$

$$\|\psi - \mathbf{E}\| \geq \left[(L+1)\binom{n}{L}(d^2-1)^L \right]^{-1/2}. \tag{19}$$

It is interesting to contrast our results with the theory of naturally fault-tolerant quantum systems proposed by Kitaev [26], and since developed by many researchers. Such systems possess a natural resilience to quantum noise processes due to their underlying physics, rather than requiring complex external control. This resilience makes them especially good candidates for quantum information processing. A feature of many naturally fault-tolerant systems is that the ground state is a quantum error-correcting code, and thus the system has the desirable property that at low temperatures it naturally sits in states of the code. Our results show that unless the code is degenerate, getting codes requires extremely nonlocal Hamiltonians that are implausible on physical grounds. Thus, the degeneracy of the quantum codes appearing in proposals for naturally fault-tolerant quantum systems is not a fluke, but rather an essential feature necessary for the system to be resilient to multiple errors.

It should be mentioned that the bound in Eq. (19) becomes trivial in the limit of very large n, i.e., for macroscopic systems. We speculate that in that limit there does not exist a state far away from eigenstates of all L-local Hamiltonians, $L \geq 2$.

To conclude, we have found interesting examples of quantum states far from the eigenstates of any nontrivial L-local Hamiltonian. Surprisingly, the states we construct can still be prepared efficiently on a quantum computer. Our construction has implications for the physics of locally interacting many-body systems, and for the theory of naturally fault-tolerant systems for quantum information processing.

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[14] For systems of dimension $d > 2$ the parameters h_σ may be complex, depending on the operator basis used in Eq. (1). However, a similar argument to that below shows that the number of independent real parameters is still given by Eq. (2), but with 3 replaced by $d^2 - 1$.
[15] We use the ground state for concreteness; the argument which follows applies equally to excited states. The fact that a Hamiltonian has many eigenstates will not affect the parameter count: we are counting real parameters, whereas the index to the particular eigenstate is a discrete parameter and thus does not change the dimension of the relevant manifold.
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