Quantum Communication through an Unmodulated Spin Chain

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We propose a scheme for using an unmodulated and unmeasured spin chain as a channel for short distance quantum communications. The state to be transmitted is placed on one spin of the chain and received later on a distant spin with some fidelity. We first obtain simple expressions for the fidelity of quantum state transfer and the amount of entanglement sharable between any two sites of an arbitrary Heisenberg ferromagnet using our scheme. We then apply this to the realizable case of an open ended chain with nearest neighbor interactions. The fidelity of quantum state transfer is obtained as an inverse discrete cosine transform and as a Bessel function series. We find that in a reasonable time, a qubit can be directly transmitted with better than classical fidelity across the full length of chains of up to 80 spins. Moreover, our channel allows distillable entanglement to be shared over arbitrary distances.

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Transmitting a quantum state (known or unknown) from one place to another is often an important task [1]. It is required, for example, to link several small quantum processors for large-scale quantum computing. Thus it is very important to have physical systems which can serve as channels for quantum communication. We can either directly transmit a state through the channel, or we can first use the channel to share entanglement with a separated party and then use this entanglement for teleportation [2]. The ideal channel for long distance quantum communications is an optical fiber. This requires interfacing a quantum computer (such as arrays of spins or ions) with optics. For short distance communications (say between adjacent quantum processors), alternatives to interfacing different kinds of physical systems are highly desirable and have been proposed, for example, for ion traps [3]. In this Letter, I propose a scheme to use a spin chain (a 1D magnet, real or simulated) as a channel for short distance quantum communication. The communication is achieved by placing a spin encoding the state at one end of the chain and waiting for a specific amount of time to let this state propagate to the other end [as shown in Fig. 1(a)]. This helps to avoid interfacing because both quantum computers and quantum channels can then be made by the same physical systems. Moreover, the spin-chain channel does not require the ability to switch "on" and "off" the interactions between the spins comprising the channel, which is often a problem in quantum computer implementations [4,5] (it, however, requires limited switching of interactions of the spin on which the initial state is encoded and the spin on which the final state is received with the rest of the chain at the start and the end of the protocol, respectively). The channel does not require any modulation by external fields (essential for quantum computation) either. This simplicity in comparison to a quantum computer makes it an ideal connector between quantum computers and realizable well before a quantum computer.

I will first present the scheme in a general setting for arbitrary graphs of spins with ferromagnetic Heisenberg interactions and later proceed to the realizable case of an open ended chain. Consider the general graph shown in Fig. 1(b), where the vertices are spins and the edges connect spins which interact. Say there are N spins in

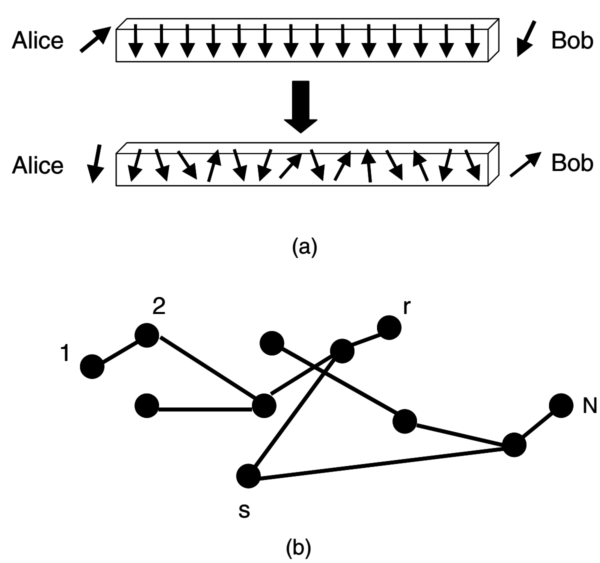

FIG. 1. (a) Our quantum communication protocol. Initially the spin chain is in its ground state in an external magnetic field. Alice and Bob are at opposite ends of the chain. Alice places the quantum state she wants to communicate on the spin nearest to her. After a while, Bob receives this state with some fidelity on the spin nearest to him. (b) An arbitrary graph of spins through which quantum communications may be accomplished using our protocol. The communication takes place from the sender spin s to the receiver spin r.

the graph and these are numbered $1, 2, \ldots, N$. The Hamiltonian is given by

$$\mathbf{H}_{\mathbf{G}} = -\sum_{\langle i,j\rangle} J_{ij}\vec{\sigma}^i.\vec{\sigma}^j - \sum_{i=1}^{N} B_i\sigma_z^i. \quad (1)$$

$\vec{\sigma}^i = (\sigma_x^i, \sigma_y^i, \sigma_z^i)$ in which $\sigma_{x/y/z}^i$ are the Pauli matrices for the ith spin, $B_i > 0$ are static magnetic fields and $J_{ij} > 0$ are coupling strengths, and $\langle i, j\rangle$ represents pairs of spins. $\mathbf{H}_{\mathbf{G}}$ describes an arbitrary ferromagnet with isotropic Heisenberg interactions. We now assume that the state sender Alice is located closest to the sth (sender) spin and the state receiver Bob is located closest to the rth (receiver) spin [these spins are shown in Fig. 1(b)]. All the other spins will be called channel spins. It is also assumed that the sender and receiver spins are detachable from the chain. In order to transfer an unknown state to Bob, Alice replaces the existing sender spin with a spin encoding the state to be transferred. After waiting for a specific amount of time, the unknown state placed by Alice travels to the receiver spin with some fidelity. Bob then picks up the receiver spin to obtain a state close to the the state Alice wanted to transfer. As we never require individual access or individual modulation of the channel spins in our protocol, they can be constituents of rigid 1D magnets.

We assume that initially the system is initially cooled to its ground state $|\mathbf{0}\rangle = |000\ldots0\rangle$ where $|0\rangle$ denotes the spin down state (i.e., spin aligned along $-z$ direction) of a spin. This is shown for a 1D chain in the upper part of Fig. 1(a). I will set the ground state energy $E_0 = 0$ (i.e., redefine $\mathbf{H}_{\mathbf{G}}$ as $E_0 + \mathbf{H}_{\mathbf{G}}$) for the rest of this Letter. We also introduce the class of states $|\mathbf{j}\rangle = |00\ldots010\ldots0\rangle$ (where $\mathbf{j} = \mathbf{1}, \mathbf{2}, \ldots \mathbf{s}, \ldots \mathbf{r}, \ldots, \mathbf{N}$) in which the spin at the jth site has been flipped to the $|1\rangle$ state. To start the protocol, Alice places a spin in the unknown state $|\psi_{\text{in}}\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle$ at the sth site in the spin chain. We can describe the state of the whole chain at this instant (time $t = 0$) as

$$|\Psi(0)\rangle = \cos\frac{\theta}{2}|\mathbf{0}\rangle + e^{i\phi}\sin\frac{\theta}{2}|\mathbf{s}\rangle. \quad (2)$$

Bob wants to retrieve this state, or a state as close to it as possible, from the rth site of the graph. Then he has to wait for a specific time until the initial state $|\Psi(0)\rangle$ evolves to a final state which is as close as possible to $\cos\frac{\theta}{2}|\mathbf{0}\rangle + e^{i\phi}\sin\frac{\theta}{2}|\mathbf{r}\rangle$. As $[\mathbf{H}_{\mathbf{G}}, \sum_{i=1}^{N}\sigma_z^i] = 0$, the state $|\mathbf{s}\rangle$ only evolves to states $|\mathbf{j}\rangle$ and the evolution of the spin graph (with $\hbar = 1$) is

$$|\Psi(t)\rangle = \cos\frac{\theta}{2}|\mathbf{0}\rangle + e^{i\phi}\sin\frac{\theta}{2}\sum_{\mathbf{j}=1}^{\mathbf{N}}\langle\mathbf{j}|e^{-i\mathbf{H}_{\mathbf{G}}t}|\mathbf{s}\rangle|\mathbf{j}\rangle. \quad (3)$$

The state of the rth spin will, in general, be a mixed state, and can be obtained by tracing off the states of all other spins from $|\Psi(t)\rangle$. This evolves with time as

$$\rho_{\text{out}}(t) = P(t)|\psi_{\text{out}}(t)\rangle\langle\psi_{\text{out}}(t)| + [1 - P(t)]|0\rangle\langle 0|, \quad (4)$$

with

$$|\psi_{\text{out}}(t)\rangle = \frac{1}{\sqrt{P(t)}}\left(\cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}f_{s,r}^N(t)|1\rangle\right), \quad (5)$$

where $P(t) = \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}|f_{r,s}^N(t)|^2$ and $f_{r,s}^N(t) = \langle\mathbf{r}|\exp\{-i\mathbf{H}_{\mathbf{G}}t\}|\mathbf{s}\rangle$. Note that $f_{r,s}^N(t)$ is just the transition amplitude of an excitation (the $|1\rangle$ state) from the sth to the rth site of a graph of N spins.

Now suppose it is decided that Bob will pick up the rth spin (and hence complete the communication protocol) at a predetermined time $t = t_0$. The fidelity of quantum communication through the channel averaged over all pure input states $|\psi_{\text{in}}\rangle$ in the Bloch sphere $[(1/4\pi)\int\langle\psi_{\text{in}}|\rho_{\text{out}}(t_0)|\psi_{\text{in}}\rangle d\Omega]$ is then

$$F = \frac{|f_{r,s}^N(t_0)|\cos\gamma}{3} + \frac{|f_{r,s}^N(t_0)|^2}{6} + \frac{1}{2}, \quad (6)$$

where $\gamma = \arg\{f_{r,s}^N(t_0)\}$. To maximize the above average fidelity, we must choose the magnetic fields B_i such that γ is a multiple of 2π. Assuming this special choice of magnetic field value (which can always be made for any given t_0) to be a part of our protocol, we can simply replace $f_{r,s}^N(t_0)$ by $|f_{N,1}^N(t_0)|$ in Eq. (5). The spin chain then acts as an *amplitude damping quantum channel* [6,7]. It converts the input state $\rho_{\text{in}} = |\psi_{\text{in}}\rangle\langle\psi_{\text{in}}|$ to $\rho_{\text{out}} = M_0\rho_{\text{in}}M_0^\dagger + M_1\rho_{\text{in}}M_1^\dagger$ with the operators M_0 and M_1 (Kraus operators [6]) given by

$$M_0 = \begin{bmatrix} 1 & 0 \\ 0 & |f_{r,s}^N(t_0)| \end{bmatrix}, \qquad M_1 = \begin{bmatrix} 0 & \sqrt{1 - |f_{r,s}^N(t_0)|^2} \\ 0 & 0 \end{bmatrix}. \quad (7)$$

Now consider the transmission of the state of one member of a pair of particles in the entangled state $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ through the channel. This is the usual procedure for sharing entanglement between separated parties which precedes teleportation [1]. The output state $\rho_{\text{out}}(t_0) = \sum_{i=0,1} I \otimes M_i|\psi^+\rangle\langle\psi^+|I \otimes M_i^\dagger$ is

$$\rho_{\text{out}}(t_0) = \frac{1}{2}\{(1 - |f_{r,s}^N(t_0)|^2)|00\rangle\langle 00| + (|10\rangle + |f_{r,s}^N(t_0)||01\rangle)(\langle 10| + |f_{r,s}^N(t_0)|\langle 01|)\}.$$

The entanglement $\mathcal{E}$ of the above state, as measured by the concurrence [8], is given by

$$\mathcal{E} = |f_{r,s}^N(t_0)|. \quad (8)$$

Thus, for any nonzero $f_{r,s}^N(t_0)$ (however small), some entanglement can be shared through the channel. This entanglement, being that of a 2×2 system, can also be distilled [9] into pure singlets and used for teleportation. Later we will estimate $f_{r,s}^N(t_0)$ for very long open chains and show that entanglement can be distributed to arbitrary distances.

Equations (6) and (8) are exceptionally simple formulas for the fidelity of quantum communications and the entanglement shared through our spin-graph channel in terms of single transition amplitude $f^N_{r,s}(t_0)$. We note here that such simple formulas, with slight modifications, will hold for spin graphs with much wider class of interactions, as long as the state $|\mathbf{0}\rangle$ does not evolve [10].

We will now consider a linear open ended spin chain [Fig. 1(a)], which is the most natural geometry for a channel. To use an analytically solvable Hamiltonian $\mathbf{H_L}$ we assume $J_{ij} = (J/2)\delta_{i+1,j}$ (nearest neighbor interactions of equal strength) and $B_i = B$ (uniform magnetic field) for all i and j in Eq. (1) for $\mathbf{H_G}$. The eigenstates of $\mathbf{H_L}$, relevant to our problem, are

$$|\tilde{m}\rangle_\mathbf{L} = a_m \sum_{j=1}^{N} \cos\left\{\frac{\pi}{2N}(m-1)(2j-1)\right\}|\mathbf{j}\rangle, \qquad (9)$$

where $m = 1, 2, \ldots, N$, $a_1 = 1/\sqrt{N}$, and $a_{m\neq 1} = \sqrt{2/N}$ with energy (on setting $E_0 = 0$) given by $E_m = 2B + 2J(1 - \cos\{\frac{\pi}{N}(m-1)\})$. In this case, $f^N_{r,s}(t_0)$ is given by

$$f^N_{r,s}(t_0) = \sum_{m=1}^{N} \langle \mathbf{r}|\tilde{m}\rangle\langle\tilde{m}|\mathbf{s}\rangle e^{-iE_m t_0} = \mathrm{IDCT}_s(v_{m,r}) \qquad (10)$$

where $v_{m,r} = a_m \cos\{\frac{\pi}{2N}(m-1)(2r-1)\}e^{-iE_m t_0}$ and $\mathrm{IDCT}_s(v_{m,r}) = \sum_{m=1}^{N} a_m v_{m,r} \cos\{\frac{\pi}{2N}(m-1)(2s-1)\}$ is the sth element of the inverse discrete cosine transform (IDCT) of the vector $\{v_{m,r}\}$.

We now want to study the performance of our protocol for various chain lengths N with $s = 1$ and $r = N$ [Alice and Bob at opposite ends of the chain as shown in Fig. 1(a)]. Bob has to wait for different lengths of time t_0 for different chain lengths N, in order to obtain a high fidelity of quantum state transfer. Using Eqs. (6), (8), and (10), I have numerically evaluated the maximum of $|f^N_{N,1}(t_0)|$ (which corresponds to the maxima of both fidelity and entanglement) for various chain lengths from $N = 2$ to $N = 80$ when Bob is allowed to choose t_0 within a finite (but long) time interval of length $T_{\max} = 4000/J$. This evaluation is fast because Eq. (10) allows us to use numerical packages for the discrete cosine transform. Taking a finite $T_{\max}$ is physically reasonable, as Bob cannot afford to wait indefinitely. It is to be understood that within $[0, T_{\max}]$, the time t_0 at which optimal quantum communication occurs varies with N. The maximum fidelities as a function of N and the maximum amounts of entanglement sharable (both rounded to three decimal places) are shown in Fig. 2.

Figure 2 shows various interesting features of our protocol. The plot also shows that in addition to $N = 2$, which is perfect (a well-known fact for the Heisenberg interaction [11]), $N = 4$ gives perfect ($F = 1.000$) quantum state transfer to three decimal places and $N = 8$ gives near perfect ($F = 0.994$). The fidelity also exceeds 0.9 for $N = 7, 10, 11, 13$, and 14. Until $N = 21$ we observe that the fidelities are lower when N is divisible by 3 in comparison to the fidelities for $N + 1$ and $N + 2$. The plot

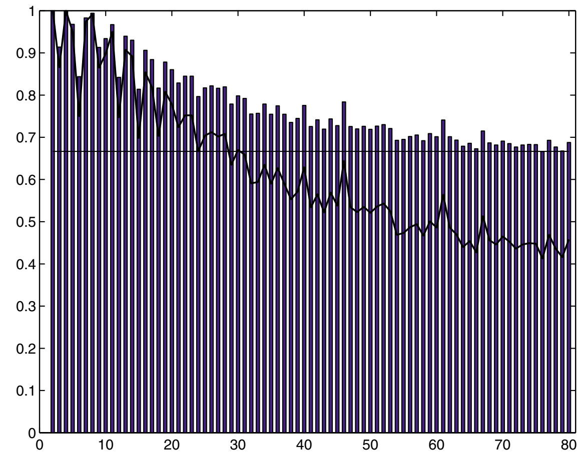

FIG. 2 (color online). The bar plot shows the maximum fidelity F of quantum communication and the curve shows the maximum sharable entanglement $\mathcal{E}$ achieved in a time interval $[0, 4000/J]$ as a function of the chain length N from 2 to 80. The time t_0 at which this maxima is achieved varies with N. The straight line at $F = 2/3$ shows the highest fidelity for classical transmission of a quantum state.

also shows that a chain of N as high as 80 exceeds the highest fidelity for classical transmission of the state, i.e., 2/3 [12] in the time interval probed by us. Of course the above results hold *only* when one is trying to directly transmit the quantum state over a distance. If one first shares entanglement through the channel, then the amount of entanglement $\mathcal{E}$ is about 0.45 for an 80 spin chain. This entanglement can be distilled to pure singlets and used for perfect teleportation.

We now estimate the entanglement sharable through chains so large that it is difficult to identify an optimal t_0 by numerical search. Hence we will choose t_0 according to a fixed (in general, nonoptimal) prescription. To motivate this choice, we expand $e^{-iE_m t_0}$ in Eq. (10) as a Bessel function series to obtain

$$\mathcal{E} = \left|\sum_{k=-\infty}^{\infty} (-1)^{Nk}[J_{N+2Nk}(\beta_0) + iJ'_{N+2Nk}(\beta_0)]\right|, \quad (11)$$

where $\beta_0 = 2Jt_0$. Using $J_N(N + \xi N^{1/3}) \approx (2/N)^{1/3} \times Ai(-2^{1/3}\xi)$ for large N [13] [where $Ai(\cdot)$ is the Airy function] we can prove that we get a maxima of $J_N(\beta_0)$ at $t_0 = (N + 0.8089N^{1/3})/2J$ and at this time

$$\mathcal{E} \approx 2|J_N(\beta_0)| \approx 1.3499N^{-1/3}, \qquad (12)$$

which ranges from 0.135 for $N = 1000$ to 1.35×10^{-4} for $N = 10^{12}$ (just 3 orders decrease in $\mathcal{E}$ for an increase in length N by 9 orders—a very efficient way to distribute entanglement). Thus for *any* finite N, however large, the chain allows us to distribute entanglement of the order $N^{-1/3}$ in a time t_0 linear in N.

As an alternate system, we now consider a ring of $2N$ spins with Hamiltonian $\mathbf{H_R}$ obtained by using

$J_{ij} = (J/2)\delta_{i\oplus 1,j}$, $B_i = B$ in Eq. (1) ($\oplus$ is summation modulo $2N$). Alice and Bob access the spins at diametrically opposite sites ($s = 1$, $r = N + 1$). In this case, the $2N$ eigenstates in the one excitation sector are $|\tilde{m}\rangle_{\mathbf{R}} = (1/\sqrt{2N})\sum_{j=1}^{2N} e^{i(\pi/v)(m-1)j}|\mathbf{j}\rangle$ and

$$\mathcal{E} = |\mathrm{IDFT}_{r-s}(u_m)| = \left|\sum_{k=-\infty}^{\infty}(-1)^{Nk}J_{N+2Nk}(\beta_0)\right|, \quad (13)$$

where $u_m = \exp(-iE_m t_0)$ and $\mathrm{IDFT}_{r-s}(u_m) = (1/2N) \times \sum_{m=1}^{2N} u_m \exp\{i(2\pi/2N)(r-s)(m-1)\}$ is the $(r-s)$th component of the inverse discrete Fourier transform of the vector $\{u_m\}$. From Eqs. (11) and (13), we numerically find that the global maxima of $\mathcal{E}$ coincide for the line and the ring. This means that by using a ring you can communicate as efficiently over a distance $r - s = N$ as you can with an open ended line over a distance $r - s = N - 1$. An immediate implication is that a four-spin ring allows perfect communication between diametrically opposite sites (because a two-spin line does [11]). This simple result in quantum information was not known until now. The coincidence of the maxima also means that the maxima of $\mathcal{E}$ for the line can also be computed by inverse Fourier transforming $\{u_m\}$.

We now mention potential systems for realization. Josephson junction arrays, excitons in quantum dots and real 1D magnets, which motivated the recent study of quantum computation in Heisenberg chains [4,5] will be good candidates. Interactions in such systems are difficult to tune. Our scheme can be implemented in such systems without the elaborate control required for quantum computation. 1D arrays of spins in solids [11,14] are also candidates. There are ring molecules described exceptionally well by $\mathbf{H}_{\mathbf{R}}$, which also allow local probes for individual spins [15](these are antiferromagnetic, but a large B could make $|\mathbf{0}\rangle$ the ground and $|\tilde{m}\rangle_{\mathbf{R}}$ the first excited states). Benzene molecules (with NMR probes possible) with $J_{ij} = (\delta_{i\oplus 1,j}/4) + (\delta_{i\oplus 2,j}/12\sqrt{3}) + (\delta_{i\oplus 3,j}/32)$, still have $|\tilde{m}\rangle_{\mathbf{R}}$ as eigenstates [16]. F can thus be calculated by an IDFT to be 0.793 for $r - s = 3$ at $t_0 = 130$. Principles of the scheme should also be testable in simulated open ended Heisenberg chains in a 1D optical lattice [17].

In this Letter, I have presented a protocol for quantum communication through an unmeasured and unmodulated spin chain. It allows quantum communication between adjacent quantum computers without interfacing different physical systems. It is well known that there exists an alternate trivial method of transferring quantum states perfectly over a distance by a series of swaps. But that requires an elaborate sequence of time dependent fields. The highly nontrivial finding of this Letter is that even without doing anything, simply by placement, quantum states can be transmitted with high fidelity over a significant distance and entanglement of the order $N^{-1/3}$ can be shared across a chain of length N. We also found that a four-spin ring allows perfect quantum communication between diametrically opposite sites. This Letter can be regarded as a study of a fundamental condensed matter system (a finite ferromagnet and its excitations) from the viewpoint of quantum communications. There remains an enormous scope for future extensions to spin graphs of varied geometry (such as in Ref. [18]) and interactions and to other well-known condensed matter systems.

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[1] C. H. Bennett and D. P. DiVincenzo, Nature (London) **404**, 247 (2000).
[2] C. H. Bennett *et al.*, Phys. Rev. Lett. **70**, 1895 (1993).
[3] D. Kielpinski, C. R. Monroe, and D. J. Wineland, Nature (London) **417**, 709 (2002).
[4] X. Zhou *et al.*, Phys. Rev. Lett. **89**, 197903 (2002).
[5] S. C. Benjamin and S. Bose, Phys. Rev. Lett. **90**, 247901 (2003).
[6] J. Preskill, http://www.theory.caltech.edu/people/preskill/ph229/.
[7] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University, Cambridge, England, 2000).
[8] W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).
[9] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. Lett. **78**, 574 (1997).
[10] This generality was pointed out to me by M. A. Nielsen.
[11] D. Loss and D. P. DiVincenzo, Phys. Rev. A **57**, 120 (1998).
[12] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. A **60**, 1888 (1999).
[13] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1972).
[14] B. E. Kane, Nature (London) **393**, 133 (1998).
[15] F. Meier and D. Loss, Phys. Rev. Lett. **86**, 5373 (2001).
[16] G. J. Bowden, J. Math. Chem. **31**, 363 (2002).
[17] L.-M. Duan, E. Demler, and M. D. Lukin, Phys. Rev. Lett. **91**, 090402 (2003).
[18] A. M. Childs *et al.*, in *Proceedings of the 35th ACM Symposium on Theory of Computing (STOC 2003), San Diego, CA* (ACM Press, New York, 2003), pp. 59–68.