Junctions of Three Quantum Wires and the Dissipative Hofstadter Model

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We study a junction of three quantum wires enclosing a magnetic flux. This is the simplest problem of a quantum junction between Tomonaga-Luttinger liquids in which Fermi statistics enter in a nontrivial way. We present a direct connection between this problem and the dissipative Hofstadter problem, or quantum Brownian motion in two dimensions in a periodic potential and an external magnetic field, which in turn is connected to open string theory in a background electromagnetic field. We find nontrivial fixed points corresponding to a chiral conductance tensor leading to an asymmetric flow of the current.

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Electric conduction in quantum wires is of much current interest. From a practical viewpoint, continuing advance in electronic technology is now reaching a level which requires understanding electric conduction in the quantum limit. From a basic science perspective, conduction in low-dimensional systems exhibits many interesting properties due to strong correlation effects. A seminal example is discussed by Kane and Fisher (KF) [1], who predicted that backscattering due to a single impurity, however small, makes the conductance vanish at low temperature in the presence of repulsive interactions. This is a clear manifestation of the non-Fermi-liquid behavior of interacting electrons in one dimension, which can generally be regarded as a Tomonaga-Luttinger liquid (TLL). A TLL is nothing but a field theory of free bosons in $1 + 1$ dimensions, and the effect of the impurity can be studied as a boundary interaction in the field theory; renormalization-group (RG) fixed points can be identified with conformally invariant boundary conditions. Study of boundary conditions in conformal field theory is an active research area in itself, with applications to string theory and statistical mechanics.

In order to make any nontrivial circuit with quantum wires, one needs a junction of three or more wires, which generalizes the KF problem. This turns out to be a surprisingly rich problem [2–8] with a number of open questions, even for three wires. One of the novel aspects of the multilead problem is that, unlike the junction of two wires (KF problem), electron fermion statistics play a crucial role [2].

Here we study a junction of three quantum wires with an enclosed magnetic flux ϕ, as shown in Fig. 1. Using a low-energy limit Dirac fermion formulation, the physical junction becomes equivalent to electron hopping between wires, represented by the boundary term

$$H_{\mathcal{B}} = -\sum_{j=1}^{3}[(\Gamma e^{i\phi/3}\psi_j^\dagger \psi_{j-1} + \text{H.c.}) + r\psi_j^\dagger \psi_j]. \quad (1)$$

Here ψ_j is the electron annihilation operator for the wire j at the junction (with $\psi_0 \equiv \psi_3$). Γ is the (real) hopping amplitude, r a backscattering potential, and ϕ is essentially the dimensionless flux through the junction (for weak hopping). r has no effect on the conductance (in the zero frequency, zero temperature limit), with the exception of the noninteracting case, so we henceforth ignore it. For simplicity, we consider spinless fermions, while considering arbitrary strength bulk interactions. In a TLL, the interaction strength is essentially contained in a single parameter g, which determines various physical quantities and exponents. $g < 1$ and $g > 1$ correspond, respectively, to repulsive and attractive interactions, while $g = 1$ is the Fermi liquid.

In the presence of a magnetic flux ϕ through the junction, the combination of the quantum phases due to the flux and fermion statistics is mapped to a "magnetic field" for the free boson field at the boundary. Such a theory has been discussed from the viewpoint of open string theory [9]. Moreover, the phase diagram in the presence of both interactions and the magnetic field at the boundary was studied in a beautiful paper by Callan and Freed (CF) [9]. We will show that a low-energy RG fixed point of our junction problem is given by the "magnetic" boundary condition of CF, which leads to a chiral conductance tensor corresponding to an asymmetric flow of the current. The magnetic boundary condition exhibits a nonmonotonic dependence on the TLL interaction

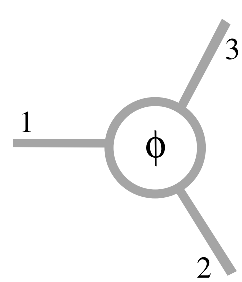

FIG. 1. A quantum junction of three wires containing a magnetic flux ϕ.

parameter g, unlike in other applications of TLL. In particular, the magnetic fixed point is stable for $1 < g < 3$. This means that an arbitrarily small magnetic flux ϕ leads to a substantial breaking of time-reversal symmetry for this range of g.

The electron annihilation operator in wire j is represented as [1]

$$\psi_j \sim \eta_j e^{i\varphi_j/\sqrt{2}}, \tag{2}$$

where η_j is the so-called Klein factor satisfying $\{\eta_j, \eta_k\} = 2\delta_{jk}$, which is necessary to ensure the anticommutation relations of the fermion operators in different wires. The Klein factors may be represented by the Pauli matrices. In the low-energy limit, the (imaginary time) effective action for the three disconnected quantum wires is given by the three component free boson

$$S = \int d\tau\, dx \sum_{j=1}^{3} \frac{g}{4\pi}(\partial_\mu \varphi_j)^2, \tag{3}$$

defined over the half-line $x > 0$, with the junction at $x = 0$. Neumann boundary conditions $\partial\varphi_j/\partial x = 0$ are satisfied.

The boundary interactions lead to renormalization to an infrared fixed point with a different boundary condition. However, current conservation at the junction requires the "center-of-mass" field

$$\Phi_0 = \frac{1}{\sqrt{3}}(\varphi_1 + \varphi_2 + \varphi_3) \tag{4}$$

to always obey Neumann boundary conditions. Thus the remaining degrees of freedom at the boundary comprise a two component boson field $\vec{\Phi} = (\Phi_1, \Phi_2)$ defined by $\Phi_1 = (\varphi_1 - \varphi_2)/\sqrt{2}$, $\Phi_2 = (\varphi_1 + \varphi_2 - 2\varphi_3)/\sqrt{6}$. Using this new basis, the hopping term is written as

$$S_{\mathcal{B}} = i\Gamma e^{i\phi/3} \int d\tau \sum_{a=1}^{3} \eta_a\, e^{i\vec{K}_a\cdot\vec{\Phi}} + \text{H.c.}, \tag{5}$$

where $\vec{K}_1 = (-1, \sqrt{3})/2, \vec{K}_2 = (-1, -\sqrt{3})/2, \vec{K}_3 = (1, 0)$. The scaling dimension of the hopping term in the disconnected limit ($\Gamma = 0$) is calculated by standard methods [1] as $1/g$, which is not affected by the Klein factors nor the magnetic flux. However, if one attempts a perturbation theory in the hopping amplitude Γ, a phase factor appears in each term. This makes the problem rather different from the standard free boson field theory, and hence it is difficult to predict the nature of the strong hopping limit. In particular, for $g > 1$, when the disconnected wire fixed point is unstable, it is difficult to identify the infrared fixed point.

Each order of the expansion of the partition function contains an integral over the correlation function

$$\langle e^{i\vec{L}_1\cdot\vec{\Phi}(\tau_1)} e^{i\vec{L}_2\cdot\vec{\Phi}(\tau_2)} \cdots e^{i\vec{L}_n\cdot\vec{\Phi}(\tau_n)}\rangle_0 = \delta_K\Big(\sum_j \vec{L}_j\Big)\exp\Big[\frac{1}{g}\sum_{j>k}\vec{L}_j\cdot\vec{L}_k \ln|\tau_j - \tau_k|^2\Big]. \tag{6}$$

where $\delta_K(\vec{L}) = 1$ if $\vec{L} = \vec{0}$ and $\delta_K(\vec{L}) = 0$ otherwise, $\vec{L}_j$ is one of the six vectors $\pm\vec{K}_{1,2,3}$, and $\langle\,\rangle_0$ denotes the expectation value with the Neumann boundary condition on $\vec{\Phi}$. Thus, identifying the vertex operator $e^{i\vec{L}_j\cdot\vec{\Phi}}$ with the displacement by $\vec{L}_j$ in the two-dimensional plane, a nonvanishing contribution (6) corresponds to a closed loop on a triangular lattice spanned by $\vec{K}_1$ and $\vec{K}_2$. The perturbation series is given by the multipoint correlation function (6) to any order, in principle. However, in the present problem, we pick up extra phase factors due to the Klein factors (and the flux ϕ.) The phase factor for a given loop is given by the product of phase factors of elementary triangular loops. Referring to the two inequivalent triangles in a triangular lattice as "up" and "down," the phase factor for the counterclockwise loop on the up triangle is determined to be

$$(-i\eta_3)e^{i\phi/3}(-i\eta_2)e^{i\phi/3}(-i\eta_1)e^{i\phi/3} = e^{i\phi}, \tag{7}$$

while that for the counterclockwise loop on the down triangle is

$$(i\eta_3)e^{-i\phi/3}(i\eta_2)e^{-i\phi/3}(i\eta_1)e^{-i\phi/3} = e^{i(\pi-\phi)}. \tag{8}$$

Namely, loops on the triangular lattice pick up a phase as if there is a staggered magnetic flux of ϕ and $\pi - \phi$ in each elementary triangle.

Fortunately, we shall see that the above perturbative expansion coincides with the "generalized Coulomb gas" studied by CF [9] in the context of the dissipative Hofstadter model (DHM) (quantum motion of a single particle under a magnetic field and a periodic potential, subject to dissipation). The "free" action reads:

$$S_0[\vec{X}] = \frac{1}{2}\int \frac{d\omega}{(2\pi)^2}[\alpha|\omega|\delta_{\mu\nu} + \beta\omega\epsilon_{\mu\nu}]X^*_\mu(\omega)X_\nu(\omega), \tag{9}$$

where $\mu, \nu = 1, 2$ and $\epsilon_{12} = -\epsilon_{21} = 1$, $\epsilon_{11} = \epsilon_{22} = 0$. α and β are related to the dissipation and the magnetic field, respectively. This determines the propagator:

$$D_{\mu\nu}(\tau) = \langle X_\mu(\tau)X_\nu(0)\rangle = -\frac{\alpha}{\alpha^2+\beta^2}\ln\tau^2\delta_{\mu\nu} + i\pi\frac{\beta}{\alpha^2+\beta^2}\,\text{sgn}\tau\,\epsilon_{\mu\nu}. \tag{10}$$

We now introduce a potential term, which is somewhat different from the "rectangular" one in Ref. [9], as

$$S_V[\vec{X}] = -Ve^{i\delta/3}\int d\tau \sum_{a=1}^{3} e^{i\vec{K}_a\cdot\vec{X}} + \text{c.c.}, \tag{11}$$

where V and δ are chosen to be real. Expanding the partition function in powers of V, a nonvanishing contribution is given by an integral over the correlation function

$$\langle e^{i\vec{L}_1\cdot\vec{X}(\tau_1)}e^{i\vec{L}_2\cdot\vec{X}(\tau_2)}\cdots e^{i\vec{L}_n\cdot\vec{X}(\tau_n)}\rangle_0 = \delta_K\Big(\sum_j \vec{L}_j\Big)\exp\Big[\frac{\alpha}{\alpha^2+\beta^2}\sum_{j>k}\vec{L}_j\cdot\vec{L}_k \ln|\tau_j-\tau_k|^2 - i\pi\frac{\beta}{\alpha^2+\beta^2}\sum_{j>k}\vec{L}_j\times\vec{L}_k \mathrm{sgn}(\tau_j-\tau_k)\Big], \tag{12}$$

where $\vec{L}_j$ is again one of the six vectors $\pm\vec{K}_{1,2,3}$.

The perturbation series in the DHM can be made to match that in the quantum wire model, including the Klein factors and flux, for appropriate choice of α and β. The absolute value of the correlation functions agree exactly if

$$\frac{\alpha}{\alpha^2+\beta^2} = \frac{1}{g}. \tag{13}$$

The phase factor for any loop in the DHM is determined by that for two elementary triangles:

$$\pi\frac{\beta}{\alpha^2+\beta^2}\vec{K}_1\times\vec{K}_2 \pm \delta = \frac{\sqrt{3}\pi}{2}\frac{\beta}{\alpha^2+\beta^2} \pm \delta, \tag{14}$$

where $+$ and $-$ signs apply to the up and down triangles, just as in the quantum wire problem, (7) and (8). Thus, if one chooses

$$\sqrt{3}\pi\frac{\beta}{\alpha^2+\beta^2} = (2n-1)\pi \tag{15}$$

with an integer n, there is an appropriate δ which makes the two expansions based on (6)–(8) and (12) coincide exactly including the phases. Because the phases are only defined modulo 2π, there is actually an infinite number of different choices of α and β labeled by the integer n. Each choice defines a quite different theory with respect to the dynamical variable $\vec{X}$, but gives the identical generalized Coulomb gas.

Now let us consider the case of $g > 1$. Then, as we have already discussed, the electron hopping is a relevant perturbation at the disconnected limit. In the DHM, for any choice of n, V is a relevant perturbation for $g > 1$. We would like to find the infrared stable fixed point reached in the low-energy limit. A simple guess is that it occurs at $V\to\infty$. The stability of the $V\to\infty$ fixed point can be determined using the instanton method [9]. Namely, in the strong potential limit, the $\vec{X}$ field is pinned at one of the minima of the potential (11). The leading perturbation to this limit is given by a tunneling between the neighboring minima, represented by an "instanton." A calculation following Ref. [9] gives the scaling dimension of such perturbation to be α times the distance between the nearest minima squared where α is determined by (13)–(15). For a generic value of ϕ, minima of the potential (11) form a triangular lattice with lattice constant $2/\sqrt{3}$. Thus the dimension of the most relevant operator at this $V\to\infty$ fixed point is

$$\Delta_n = \frac{4g}{3+(2n-1)^2g^2}. \tag{16}$$

This implies that the $V\to\infty$ fixed point is unstable for any value of g for all choices of n except for $n = 0$ or 1, where it is stable for $1 < g < 3$. In this range of g we expect that the infrared limit of the junction model corresponds to the $V\to\infty$ limit of the DHM for $n = 0$ or 1. The infrared fixed point for other values of n must correspond to nontrivial intermediate V fixed points, for which we do not know explicit solutions. The infrared fixed point for each value of n should give the same physical behavior for the junction model. In principle we could study this using any choice of n. However, clearly the $n = 0$ or 1 choices are preferred because only in those cases can we explicitly analyze the fixed points. Let us denote the RG fixed points corresponding to the strong potential limit in these cases as χ_- and χ_+, respectively, for $n = 0$ and 1. Since $\chi_\pm$ have the same stability, at this stage we cannot determine which of these two fixed points is realized in the infrared limit for a generic value of ϕ.

On the other hand, in the special cases $\phi = \pm\pi/2$ which maximally break the time-reversal symmetry, the choice of the fixed point is unique. For $\phi = \pi/2$, the choice $n = 1$ (β positive) gives $\delta = 0$, and the potential minima forms a triangular lattice as usual. However, the other choice $n = 0$ (β negative) gives $\delta = \pi$, for which the potential minima form a honeycomb lattice with minimum distance $2/3$, making the strong potential limit unstable for all g. Similarly, for $\phi = -\pi/2$, $n = 0$ (β negative) is the unique choice to give a stable fixed point. This suggests that these fixed points $\chi_\pm$ reflect the breaking of time-reversal symmetry due to the magnetic flux ϕ. Indeed, the conductance at these fixed points exhibits a chiral behavior breaking the time-reversal invariance, as we show below. On the other hand, changing the flux corresponds to an irrelevant perturbation at these fixed points. Thus we conjecture that the flow from small V goes to the $V\to\infty$ fixed point in the $n = 1$ representation, but not in the $n = 0$ representation, where we expect the flow to be to a nontrivial fixed point, for $0 < \phi < \pi$. Conversely, for $-\pi < \phi < 0$, we conjecture that the infrared fixed point is given by the $V\to\infty$ one in the $n = 0$ representation, not $n = 1$.

The nature of the $\chi_\pm$ fixed points is reflected in the conductance at the junction, which is an experimentally measurable quantity. The conductance is expressed as a tensor

$$I_j = \sum_k G_{jk}V_k, \tag{17}$$

where I_j is the total current flowing into the junction from wire j and V_k is the voltage applied to wire k. From current conservation $\sum_j G_{jk} = 0$. It is also clear that the currents depend only on voltage differences. According to the Kubo formula, the conductance tensor is given by

$$G_{jk} = -\lim_{\omega\to 0^+} \frac{1}{\omega} \frac{\partial^2 \ln Z}{\partial A_j(-\omega)\partial A_k(\omega)}, \tag{18}$$

where ω is the Matsubara frequency, and A_j and V_j are related in real time t by $dA_j/dt = -V_j$. Assuming that the voltage drops only across the junction, its effect can be represented as an extra time-dependent phase factor (vector potential) $e^{i(A_j - A_k)}$ in the electron hopping term from wire k to wire j. It can be conveniently included in the CF representation as

$$S_V[\vec{X}] = -Ve^{i\delta/3} \int d\tau \sum_{j=1}^{3} e^{i\vec{K}_j\cdot(\vec{X}+\vec{a})} + \text{c.c.}, \tag{19}$$

where $\vec{a} = \sum_{j,k,l} \vec{K}_j \epsilon_{jkl}(A_k - A_l)/3$. Thus the conductance is related to

$$\frac{\partial^2 \ln Z}{\partial A_j \partial A_k} = \frac{4}{9} \sum_{m,n,\mu,\nu} \epsilon_{jm}\epsilon_{kn} K_m^\mu K_n^\nu \frac{\partial^2 \ln Z}{\partial a^\mu \partial a^\nu}, \tag{20}$$

where $\epsilon_{12} = \epsilon_{23} = \epsilon_{31} = 1$, $\epsilon_{21} = \epsilon_{32} = \epsilon_{13} = -1$, and $\epsilon_{jj} = 0$, $\mu, \nu = 1, 2$ refers to the components of two-dimensional vectors $\vec{a}$ and $\vec{K}_j$. In fact,

$$\frac{\partial^2 \ln Z}{\partial a^\mu(-\omega)\partial a^\nu(\omega)} = -\langle \rho_\mu(\omega)\rho_\nu(-\omega)\rangle \tag{21}$$

is the correlation function of the generalized Coulomb gas density as defined in Ref. [9], which we use to determine the conductance tensor. In the strong potential limit of the CF representation, the $\vec{X}$ field is pinned and the correlation function is given exactly by

$$\langle \rho_\mu(\omega)\rho_\nu(-\omega)\rangle = \alpha|\omega|\delta_{\mu\nu} + \beta\omega\epsilon_{\mu\nu}. \tag{22}$$

From (18)–(22), we obtain the conductance tensor

$$G_{jk}^{\pm} = G[(3\delta_{jk} - 1) \pm g\,\epsilon_{jk}]/2, \tag{23}$$

with $G = (e^2/h)4g/(3 + g^2)$ and $\pm$ corresponding to the fixed points $\chi_\pm$. The term proportional to ϵ_{jk} represents an asymmetric conduction which is allowed by the Z_3 symmetry of the Hamiltonian once time-reversal symmetry is broken by the flux.

For example, when a voltage $V_1 > 0$ is applied to wire 1 while $V_2 = V_3 = 0$, the current flowing from wire 1 to the junction is $I_1 = GV_1$, which is independent of the asymmetry. On the other hand, the currents flowing out from the junction to wires 2 and 3 are given by $-I_2 = G(1 \pm g)V_1/2$ and $-I_3 = G(1 \mp g)V_1/2$. In fact, for $g > 1$, the current for fixed point χ_+ (χ_-) rather *flows in towards the junction* from wire 3 (wire 2), opposite to the voltage drop, when the voltage $V_1 > 0$ is applied only on wire 1.

The asymmetric conduction is forbidden under time reversal but is possible in the presence of magnetic flux ϕ. In the limit of $g \to 1$, which corresponds to noninteracting electrons, we find, for example, $G_{11} = -G_{12} = 1$, $G_{13} = 0$ for χ_+. This asymmetric conduction of noninteracting electrons can be simply understood as a boundary condition $\psi_j^{\text{out}} = \psi_{j-1}^{\text{in}}$. At $g = 1$, there is in fact a continuous family of boundary conditions characterized by a 3×3 S matrix S as $\psi_j^{\text{out}} = \sum_k S_{jk}\psi_k^{\text{in}}$. Our $\chi_\pm$ correspond to special cases of these free fermion boundary conditions in the $g \to 1$ limit.

A remarkable aspect of the fixed points $\chi_\pm$ is that the conductance (23), as well as the scaling dimension (16), exhibits nonmonotonic dependence on the interaction parameter g, unlike in other known applications of TLL. Making the electron interaction more attractive can decrease the conductance.

In summary, we studied a junction of three quantum wires of interacting spinless electrons with a magnetic flux. The fermionic statistics, together with the magnetic flux, bring nontrivial phases into the problem, making it different from the standard boundary problem of free boson field theory. It is shown to be equivalent to a certain generalized Coulomb gas introduced by CF. Using the mapping, we have shown that the chiral fixed points $\chi_\pm$ exhibit an asymmetric conduction and nonmonotonic dependence on the interaction parameter g. The fixed points $\chi_\pm$ are stable for $1 < g < 3$, and we expect the junction with $\phi = \pi/2$ ($\phi = -\pi/2$) to be renormalized to the fixed point χ_+ (χ_-), since it is the unique stable fixed point obtained with the CF representation at $V \to \infty$. We conjecture that the system renormalizes to these fixed points (for $1 < g < 3$) for all values of ϕ except those which respect time reversal: ϕ/π integer.

We note in passing that our results also apply to an equivalent problem if we interpret $\vec{\Phi}$ as the coordinate of a neutral spin-1/2 particle (3) and (5), describe its propagation in a periodic textured Zeeman field with components $B_a(\vec{\Phi}) = 4\Gamma \sin(\vec{K}_a \cdot \vec{\Phi} + \phi/3)$, and Hamiltonian $H_Z = -\vec{B}(\vec{\Phi}) \cdot \vec{S}$, subject to dissipation.

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