Comment on "Universal Behavior of Load Distribution in Scale-Free Networks"

In a previous Letter [1], Goh *et al.* have presented a numerical study of the load—or betweenness centrality—distribution in a scale-free network whose degree distribution follows a power law $p(k) \sim k^{-\gamma}$ where $\gamma \in [2, \infty[$ is a tunable parameter. They showed that the load ℓ is distributed according to a power law $P(\ell) \sim \ell^{-\delta}$ with exponent δ. On the basis of their numerical results, they conjectured that the value of $\delta \simeq 2.2$ is independent of γ for the interval $[2, 3)$. Based on this apparent universality, a classification of scale-free networks according to the value of $\delta \simeq 2.2$ (class I) or $\delta = 2$ (class II) was proposed [2]. In this Comment we argue that the value of δ is not universal and varies significantly as γ changes in the interval $[2, 3)$. The power law fits of the cumulative function Prob(load $\geq \ell$) for the model proposed in [1] gives the values $\delta = 1.84 \pm 0.04$, 2.05 ± 0.05, and 2.25 ± 0.05 for $\gamma = 2$, 2.5, and 3, respectively, while for the Barabási-Albert (BA) model [3]$\delta = 2.3 \pm 0.1$. The variations of δ are significant enough to claim that it is not universal, but in order to double check our results we use an indirect way of computing δ. We study the relation between the load and the connectivity [1,4] which is of the form $\ell \sim k^{\eta}$, where the exponent η depends on the network. As can be seen in Fig. 1(a), the power law holds remarkably for a large range of k, allowing for an accurate measure of η. We also checked that the value of η does not change significantly for different values of the system size. (For $\gamma = 2.5$, we obtain a relative variation due to size going from $N = 10^4$ to 5×10^4 less than 1%.) The exponents η and δ are not independent, and it is easy to show that [4] $\eta = (\gamma - 1)/(\delta - 1)$. If the value of $\delta \simeq 2.2$ is universal, then η is a linear function of γ with slope $\simeq 1/1.2 \simeq 0.83$. In Fig. 1(b) we plot the measured η versus γ for the different types of networks studied and the corresponding value predicted by universality. Figure 1(b) shows that if for $\gamma \simeq 3$ the value $\delta = 2.2$ seems to be acceptable, the claim of universality for $\gamma \in [2, 3)$ proposed in [1] does not hold (our results do not fit in the other class $\delta = 2.0$ either). In addition, we tested the universality for different values of m, and we also obtain variations ruling it out: For $\gamma = 2.5$ and for $N = 2 \times 10^4$, we obtain $\eta = 1.477 \pm 0.006$, 1.56 ± 0.006, and 1.64 ± 0.01 for $m = 2$, 4, and 6, respectively.

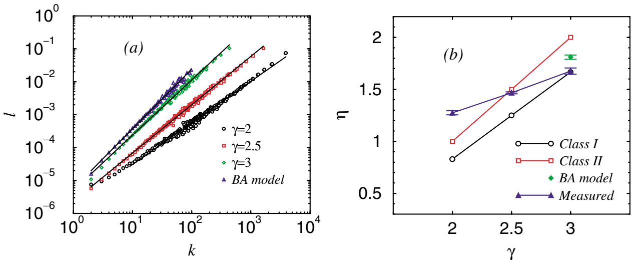

FIG. 1 (color online). (a) Log-log plot of the normalized average load versus connectivity for the same models as in [1] with $m = 2$. The power law fits (straight lines) give $\eta = 1.27 \pm 0.01$ ($N = 3 \times 10^4$), 1.467 ± 0.006 ($N = 5 \times 10^4$), and 1.68 ± 0.02 ($N = 5 \times 10^4$) for $\gamma = 2$, 2.5, and 3, respectively. For the BA model, $\eta = 1.81 \pm 0.02$ ($N = 5 \times 10^4$). (b) η vs γ. If the universality proposed in [1] would be correct, the measured values for $\gamma \in [2, 3)$ should lie on the "universal" straight line corresponding to $\delta = 2.2$ (class I).

The important exponent thus appears to be η, and it is interesting to note that η is significantly smaller than the maximum value $\eta = 2$. This maximum value is reached when nodes with large centrality (i.e., with large k) link together disconnected parts of roughly the same size. The load of these nodes is then of the order of $\ell \sim k(k - 1)/2 \sim k^2$. The fact that $\eta < 2$ indicates that the different parts are also connected by shortest paths which do not pass through the central node. More generally, it would be interesting to understand how η depends on the different parameters of the network such as γ and the degree correlation.

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[1] K.-I. Goh, B. Kahng, and D. Kim, Phys. Rev. Lett. **87**, 278701 (2001).
[2] K.-I. Goh, H. Jeong, B. Kahng, and D. Kim, Proc. Natl. Acad. Sci. U.S.A. **99**, 12 583 (2002).
[3] A.-L. Barabási and R. Albert, Science **286**, 509 (1999).
[4] A. Vazquez, R. Pastor-Satorras, and A. Vespignani, Phys. Rev. E **65**, 066130 (2002).

