Communication Cost of Simulating Bell Correlations

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What classical resources are required to simulate quantum correlations? For the simplest and most important case of local projective measurements on an entangled Bell pair state, we show that exact simulation is possible using local hidden variables augmented by just one bit of classical communication. Certain quantum teleportation experiments, which teleport a single qubit, therefore admit a local hidden variables model.

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Recent theoretical research into quantum algorithms [1], quantum communication complexity [2], and quantum cryptography [3] has shown that quantum devices are more powerful than their classical counterparts. Indeed, the flourishing field of quantum information theory [4] aims to provide an information-theoretic quantification of the power underlying quantum resources. One important feature of quantum theory lies in the statistical correlations produced by measurements on local components of a quantum system. Almost 40 years ago, Bell showed that such correlations cannot be explained by descriptions based on realistic properties of local subsystems [5]. To experimentally distinguish quantum correlations from those produced by local hidden variables theories, Bell introduced the notion of *Bell inequalities,* with subsequent experimental evidence falling squarely on the side of quantum theory [6]. Bell inequalities, while usually considered relevant only to foundational studies of quantum theory, answer a fundamental information-theoretic question: what correlations can be produced between separate classical subsystems, which have interacted in the past, if no communication between the subsystems is allowed? Violation of a Bell inequality, however, does nothing to *quantify* what classical information-processing resources are required to simulate a particular set of quantum correlations.

The simplest and most important example of quantum correlations involves the correlations produced by projective measurements on a Bell pair. Bell pairs are the maximally entangled states of two quantum bits (qubits) and are the basic resource currency of bipartite quantum information theory. Various equivalences are known: one shared Bell pair plus two bits of classical communication can be used to teleport one qubit [7] and, conversely, one shared Bell pair plus a single qubit of communication can be used to send two bits of classical communication via superdense coding [8].

Consider the gedanken experiment of Einstein, Podolsky, and Rosen (EPR) [9], as reformulated by Bohm [10]. Two spatially separate parties, Alice and Bob, each have a spin-$\frac{1}{2}$ particle, or qubit. The global spin wave function is the entangled Bell singlet state (also known as an EPR pair) $|\psi\rangle = (1/\sqrt{2})(|\uparrow\rangle_A|\downarrow\rangle_B - |\downarrow\rangle_A|\uparrow_B\rangle)$. The spin states $|\uparrow\rangle$, $|\downarrow\rangle$ are defined with respect to a local set of coordinate axes: $|\uparrow\rangle$ ($|\downarrow\rangle$) corresponds to spin along the local $+\hat{z}$ ($-\hat{z}$) direction. Alice and Bob each measure their own particle's spin along a direction parametrized by a three-dimensional unit vector: Alice measures along $\hat{a}$, Bob along $\hat{b}$. Alice and Bob obtain results, $\alpha \in \{+1, -1\}$ and $\beta \in \{+1, -1\}$, respectively, which indicate whether the spin was pointing along $(+1)$ or opposite (-1) the direction each party chose to measure. Locally, Alice and Bob's outcomes appear random, with expectation values $\langle\alpha\rangle = \langle\beta\rangle = 0$, but their joint probabilities are correlated such that $\langle\alpha\beta\rangle = -\hat{a}\cdot\hat{b}$. We refer to these correlations as *Bell correlations.*

It is not possible to reproduce these correlations using a protocol which draws on random variables shared between Alice and Bob, but does not allow communication after they have selected measurements [5]. So how much communication *is* required to exactly simulate them [11–17]? Naively, Alice can just tell Bob the direction of her measurement $\hat{a}$ (or vice versa) but this requires an infinite amount of communication. The question of whether a simulation can be done with a finite amount of communication was raised independently by Maudlin [11], Brassard, Cleve, and Tapp [12], and Steiner [13]. Their approaches differ in how the communication cost of the simulation is defined: Brassard *et al.* take the cost to be the number of bits sent in the worst case, and Steiner, the average. (Steiner's model is weaker because the amount of communication in the worst case can be unbounded, although such cases occur with probability zero.) Brassard *et al.* present a protocol which simulates Bell correlations using exactly eight bits of communication (since improved to six bits [14]). Surprisingly [15], the only lower bound for the amount of communication is given by Bell's theorem: at least some communication is needed. Here we present a simple protocol that uses just one bit of communication.

We first note three simple properties of Bell correlations: (i) if $\hat{a} = \hat{b}$, then we must have $\alpha = -\beta$: Alice and

Bob must output perfectly anticorrelated bits; (ii) either party can reverse its measurement axis and flip its output bit; and (iii) the joint probability is dependent only on $\hat{a}$ and $\hat{b}$ via the combination $\hat{a} \cdot \hat{b}$. In his original paper, Bell gave a local hidden variables model that reproduces these three properties for all possible axes, but his model fails to reproduce Bell correlations because the statistical correlations when $\hat{a} \neq \hat{b}$ are not as strong as those of quantum mechanics [5]. The protocol we describe below is inspired by Bell's original protocol. Property (iii) implies that we may restrict attention to rotationally invariant protocols, for which all probabilities depend only on $\hat{a} \cdot \hat{b}$ and not $\hat{a}$ and $\hat{b}$ separately, by *randomizing over all inputs with the same dot product.* More precisely, suppose P is any protocol that simulates the correlations. Then define a new protocol P' whose hidden variables consist of (i) those required by P, and (ii) a random rotation $R \in SO(3)$. Protocol P' then consists of running protocol P on $R\hat{a}$ and $R\hat{b}$ in place of $\hat{a}$ and $\hat{b}$.

We now describe our protocol. Alice and Bob share two random variables $\hat{\lambda}_1$ and $\hat{\lambda}_2$ which are real three-dimensional unit vectors. They are chosen independently and distributed uniformly over the unit sphere.

The protocol proceeds as follows: (1) Alice outputs $\alpha = -\text{sgn}(\hat{a} \cdot \hat{\lambda}_1)$. (2) Alice sends a single bit $c \in \{-1, +1\}$ to Bob where $c = \text{sgn}(\hat{a} \cdot \hat{\lambda}_1)\text{sgn}(\hat{a} \cdot \hat{\lambda}_2)$. (3) Bob outputs $\beta = \text{sgn}[\hat{b} \cdot (\hat{\lambda}_1 + c\hat{\lambda}_2)]$, where we have used the sgn function defined by $\text{sgn}(x) = +1$ if $x \geq 0$ and $\text{sgn}(x) = -1$ if $x < 0$. A geometric description of our protocol is given in Fig. 1. We note immediately that Bob obtains *no information* about Alice's output from the communication.

We now prove the protocol reproduces the correct expectation values. Each party's output changes sign under the symmetry $\hat{\lambda}_1 \leftrightarrow -\hat{\lambda}_1$, $\hat{\lambda}_2 \leftrightarrow -\hat{\lambda}_2$, so $\langle \alpha \rangle = \langle \beta \rangle = 0$ because $\hat{\lambda}_1$ and $\hat{\lambda}_2$ are uniformly distributed. The joint expectation value $\langle \alpha\beta \rangle$ can be calculated using

$$\langle \alpha\beta \rangle = E\left\{ -\text{sgn}(\hat{a} \cdot \hat{\lambda}_1) \sum_{d=\pm 1} \frac{(1 + cd)}{2} \times \text{sgn}[\hat{b} \cdot (\hat{\lambda}_1 + d\hat{\lambda}_2)] \right\}, \quad (1)$$

where $E\{x\} = \frac{1}{(4\pi)^2} \int d\hat{\lambda}_1 \int d\hat{\lambda}_2\, x$, $c = \text{sgn}(\hat{a} \cdot \hat{\lambda}_1)\,\text{sgn}(\hat{a} \cdot \hat{\lambda}_2)$ and we have used the trick that $(1 + cd)/2 = 1$ if $c = d$ and 0 if $c \neq d$. After substituting for c and expanding Eq. (1), we obtain the sum of four terms (because each term inside the summation sign is itself the sum of two terms) and, using $\text{sgn}(\hat{a} \cdot \hat{\lambda}_1)\, c = \text{sgn}(\hat{a} \cdot \hat{\lambda}_2)$, we note that the four terms are related by the symmetries $\hat{\lambda}_1 \leftrightarrow \hat{\lambda}_2$ or $\hat{\lambda}_2 \leftrightarrow -\hat{\lambda}_2$, so each has the same expectation value. Hence

$$\langle \alpha\beta \rangle = E\{2\,\text{sgn}(\hat{a} \cdot \hat{\lambda}_1)\,\text{sgn}[\hat{b} \cdot (\hat{\lambda}_2 - \hat{\lambda}_1)]\}. \quad (2)$$

This integral may be evaluated with the help of the two

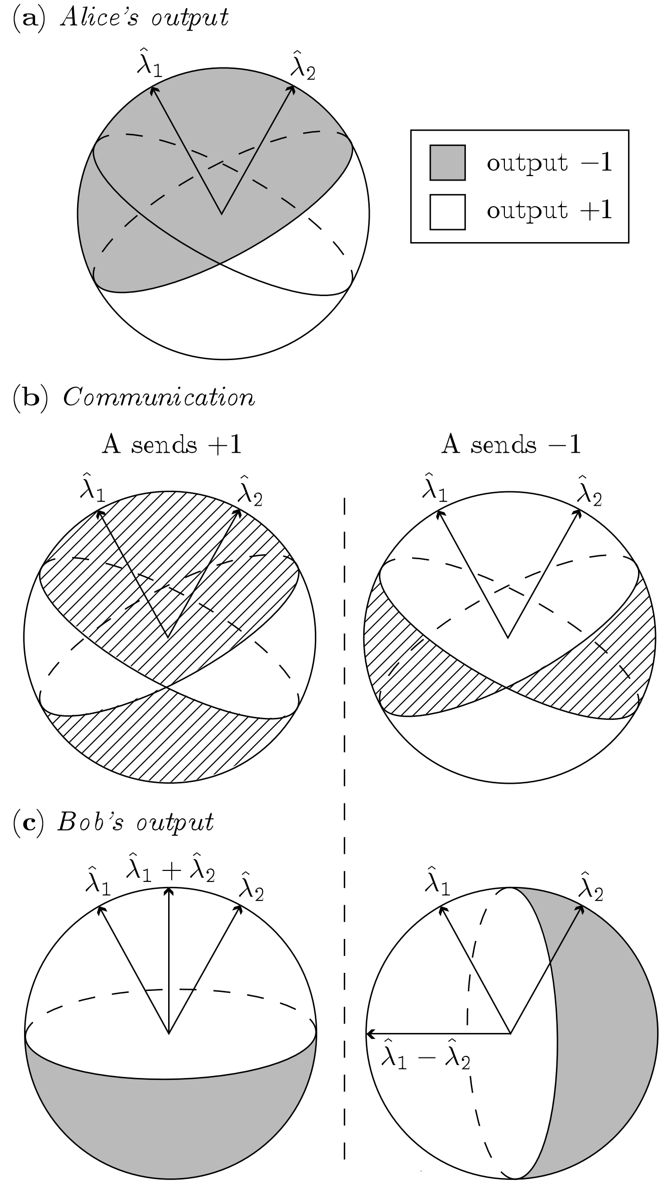
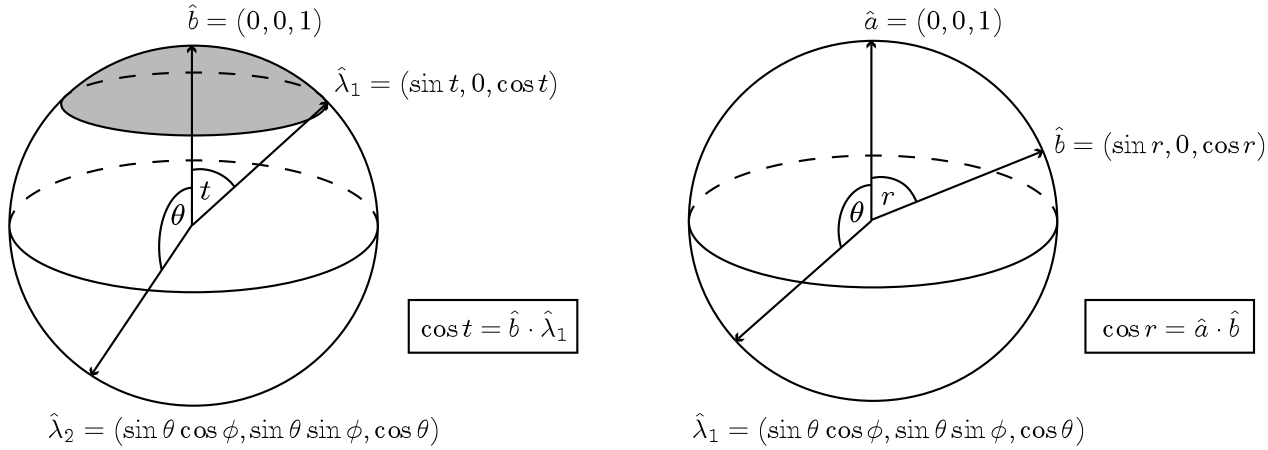


FIG. 1. *The protocol:* The shared unit vectors $\hat{\lambda}_1$ and $\hat{\lambda}_2$ described in the text divide the Bloch sphere into four quadrants, as shown. Alice and Bob's actions depend on which quadrant their respective measurement axes lie in, and in Bob's case, the bit he receives from Alice. (a) *Alice's output:* if $\hat{a}$ lies in the shaded region, Alice outputs −1; in the unshaded region, she outputs +1. (b) *The communication:* Alice sends $c = +1$ if her measurement axis lies in the N or S quadrants, and −1 otherwise. (c) *Bob's output:* this depends on the bit received from Alice. The shading is as for (a).

diagrams shown in Fig. 2, with the result that $\langle \alpha\beta \rangle = -\hat{a} \cdot \hat{b}$, as required.

Our protocol exactly simulates the quantum mechanical probability distribution for projective measurements on the singlet Bell pair state. If a large number of simulations are performed in parallel, the communication may be compressed. To see this, assume Alice's measurement vector $\hat{a}$ is uniformly distributed (if not, we



(a) $\frac{1}{4\pi}\int d\hat{\lambda}_2 \,\mathrm{sgn}\left[\hat{b}\cdot(\hat{\lambda}_2 - \hat{\lambda}_1)\right] = -\hat{b}\cdot\hat{\lambda}_1$ (b) $\frac{1}{2\pi}\int d\hat{\lambda}_1 \,\mathrm{sgn}(\hat{a}\cdot\hat{\lambda}_1)\hat{b}\cdot\hat{\lambda}_1 = \hat{a}\cdot\hat{b}$

FIG. 2. Construction used to evaluate Eq. (2): (a) We first integrate over $\hat{\lambda}_2$, taking $\hat{b}$ to point along the positive z axis [18]. Observe that $\mathrm{sgn}[\hat{b}\cdot(\hat{\lambda}_2 - \hat{\lambda}_1)]$ is positive in the top spherical cap (shaded) and negative otherwise. The area of the top spherical cap is $A_+ = 2\pi\int_0^t \sin\theta d\theta = 2\pi(1-\cos t)$ where $\cos t = \hat{b}\cdot\hat{\lambda}_1$, hence $\int d\hat{\lambda}_2 \,\mathrm{sgn}[\hat{b}\cdot(\hat{\lambda}_2 - \hat{\lambda}_1)] = A_+ - (4\pi - A_+) = -4\pi\cos t = -4\pi\hat{b}\cdot\hat{\lambda}_1$. (b) We now take $\hat{a}$ to point along the positive z axis [19], set $\hat{b} = (\sin r, 0, \cos r)$, and integrate over $\hat{\lambda}_1$, obtaining $\int d\hat{\lambda}_1 \,\mathrm{sgn}(\hat{a}\cdot\hat{\lambda}_1)\hat{b}\cdot\hat{\lambda}_1 = \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi \,\mathrm{sgn}(\cos\theta)(\cos\theta\cos r + \sin\theta\cos\phi\sin r) = 2\pi\cos r = 2\pi\hat{a}\cdot\hat{b}$.

randomize, as outlined above). Then, if $\hat{\lambda}_1\cdot\hat{\lambda}_2 = \cos\eta$, Alice sends -1 with probability η/π and 1 with probability $1-\eta/\pi$, so that the communication can be compressed to $\int_0^{\pi/2}\sin\eta\, d\eta H(\eta/\pi) \approx 0.85$ bits, where $H(\eta/\pi)$ is the Shannon entropy. This encoding depends on the shared unit vectors $\hat{\lambda}_1$ and $\hat{\lambda}_2$: a third party without access to the hidden variables will observe Alice sending uniformly distributed bits to Bob.

Our protocol is easily modified to simulate joint measurements on any maximally entangled state of two qubits, because every such state is related to the singlet by a local change of basis and thus may be simulated by rotating and/or reflecting the input vectors $\hat{a}$ and $\hat{b}$, before running our protocol.

Now consider the following experiment: Alice prepares a qubit in a state unknown to Bob. She then teleports the qubit to Bob, who performs a projective measurement on it, along a direction unknown to Alice. We shall show that this experiment admits a local hidden variables description. We first note that quantum teleportation experiments do not purport to test whether quantum mechanics allows a local hidden variables model; rather they aim to distinguish quantum teleportation from other protocols Alice and Bob might carry out using classical communication, but no entanglement [20]. From this point of view, teleportation experiments represent "investigations *within* quantum mechanics" [21], rather than comparisons of quantum mechanics with classical local hidden variables models [22]. With this distinction in mind, it is still interesting to ask whether teleportation experiments *can* be explained by a local hidden variables model.

If one allows an infinite amount of classical communication from Alice to Bob, then there is a trivial local hidden variables model, for Alice can just send a classical description of the state to Bob, who then simulates his measurement. We now give a local hidden variables model that requires only two bits of communication, which is the *same* amount as the quantum teleportation protocol. The construction is based on Ref. [16], where the procedure is termed "classical teleportation." It is sufficient to consider the case where Alice prepares the qubit in a pure state, which we suppose has spin aligned along the axis $\hat{a}$. We suppose Bob's measurement is aligned along the axis $\hat{b}$. Alice and Bob share uniformly distributed random three-dimensional unit vectors $\hat{\lambda}_1$ and $\hat{\lambda}_2$ (which can be thought of as hidden variables carried by the Bell pair used for teleportation). The protocol is as follows: (1) Alice sends $c_1 = \mathrm{sgn}(\hat{a}\cdot\hat{\lambda}_1)$ and $c_2 = \mathrm{sgn}(\hat{a}\cdot\hat{\lambda}_2)$ to Bob. (2) Bob outputs $\beta = \mathrm{sgn}[\hat{b}\cdot(c_1\hat{\lambda}_1 + c_2\hat{\lambda}_2)]$. It is easy to verify that $\langle\beta\rangle = \hat{a}\cdot\hat{b}$, as required. We also note that the two bits sent appear completely random to a party without access to the hidden variables.

It is usual in teleportation experiments to have (i) a third party Victor supply Alice with a quantum state unknown to her, and (ii) Bob hand off the teleported state to Victor (or another party) to measure, rather than measuring it himself. Such a distinction is not important for the question we address, because the qubit transmitted from Victor to Alice, for example, can carry hidden variables describing its state. The point is that local hidden variables are *hidden:* although it is convenient to describe a local hidden variables model as if Alice and Bob had access to the hidden variables, the model still exists even if the hidden variables are inaccessible to them. There is no way for the experimenters to tell whether their experiment is described by quantum theory or by "gremlins" within their apparatus, executing the local hidden variables protocol described above.

Are there quantum teleportation experiments which do not have such a local hidden variables description? One obvious possibility is an experiment that teleports entanglement itself. But there is a more subtle possibility. If we allow Bob to measure the qubit using elements of a positive operator-valued measure, then there may not be a local hidden variables description which respects the two bit classical communication bound. More generally, if Alice teleports n qubits (which requires $2n$ bits of communication) and Bob makes a joint measurements on them, then it is known that any exact local hidden variables theory requires that Alice send at least a constant times 2^n bits of communication in the worst case [12]. Whether this holds for protocols with bounded error is an important open question.

Finally, using the classical teleportation protocol, we obtain a (not necessarily optimal) protocol to simulate joint projective measurements on partially entangled states of two qubits, which uses two bits of communication: Alice first simulates her measurement and determines the postmeasurement state of Bob's qubit; Alice and Bob then execute the classical teleportation protocol.

The results presented here offer an intriguing glimpse into the nature of correlations produced in quantum theory. If we interpret Bell inequality violation to mean that some communication is necessary to simulate Bell correlations, then our results prove that the minimal amount, one bit, is all that is necessary for projective measurements on Bell pairs. Is our straightforward protocol an indication of a deep structure in quantum correlations? We hope that our protocol and the development of a general theory of the communication cost of simulating quantum correlations will help shed light on this fundamental question.

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