Inflation versus Cyclic Predictions for Spectral Tilt

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We present a nearly model-independent estimate that yields the predictions of a class of simple inflationary and ekpyrotic or cyclic models for the spectral tilt of the primordial density inhomogeneities that enables us to compare the two scenarios. Remarkably, we find that the two produce an identical result, $n_s \approx 0.95$. For inflation, the same estimate predicts a ratio of tensor to scalar contributions to the low l multipoles of the microwave background anisotropy of $T/S \approx 20\%$; the tensor contribution is negligible for ekpyrotic or cyclic models, as shown in earlier papers.

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The recent measurement of the cosmic microwave background (CMB) anisotropy by the Wilkinson Microwave Anisotropy Probe (WMAP) [1] is consistent with a primordial power spectrum of density fluctuations that is scale invariant, Gaussian, and adiabatic. These characteristics coincide with the predictions of the simplest inflationary scenarios [2].

In this Letter, we show that these are also predicted by the simplest ekpyrotic or cyclic scenarios [3–5]. We compare the density fluctuation spectra obtained in inflationary [2] and ekpyrotic or cyclic models by computing their predictions for an important, well-motivated class of simple models. We find surprisingly similar predictions for the spectral index of the scalar density fluctuations [6]. Both predict a red spectrum with index $n_s \approx 0.95$. For inflation, the same argument predicts a ratio of tensor to scalar contributions to the low l multipoles of roughly 20%. Our results for inflation are not new; the particular form of the argument presented here is a variant of the discussion in Ref. [7] and by Mukhanov [8] and gives a similar result to other estimates. But our result for the ekpyrotic or cyclic models and the similarity to the inflationary prediction is both new and unexpected.

For the ekpyrotic or cyclic models, scale-invariant fluctuations are generated during a period of slow contraction. The notion is that these imprint themselves as temperature fluctuations in the current expanding phase [9]. The validity of this idea has been debated [9–11], with different answers obtained depending on assumptions about the precise matching conditions at the bounce. Here we use the recent results of Tolley *et al.* [12], which treat the bounce as a collision of branes in five dimensions, derive a unique matching condition, and find a scale-invariant spectrum of temperature fluctuations after the bounce.

Both inflation and the ekpyrotic or cyclic models rely on the equation of state parameter w having a specific qualitative behavior throughout the period when fluctuations are generated, including the interval when fluctuations with wavelengths within the present horizon radius were produced (corresponding to the last $\mathcal{N} \approx 60$ e-folds in wavelength). For inflation, the condition on w is that $1 + w \ll 1$ and for ekpyrotic or cyclic models it is $w \gg 1$ [9,13]. Correspondingly, the Hubble constant H is nearly constant during inflation and the four dimensional scale factor a is nearly constant during ekpyrosis. Since these conditions must be maintained for the duration of an epoch spanning many more than $\mathcal{N}$ e-folds, the simplest possibility is to suppose that w (and correspondingly H or a) change slowly and monotonically during that last $\mathcal{N}$ e-folds. More precisely, we take "simplest" to mean that (i) $dw/d\mathcal{N}$ is small, and $d^2w/d\mathcal{N}^2$ or $(dw/d\mathcal{N})^2$ negligible and (ii) in order for inflation (ekpyrosis) to end, H during inflation (or a during ekpyrosis) decays by a factor of order unity over the last $\mathcal{N}$ e-folds. Tilts or spectral features that differ from those presented here can be produced only by introducing by hand unnecessary rapid variations in w—unnecessary in the sense that they are not required for either model to give a successful account of the standard cosmology.

Note that our condition on the time variation of w does not refer directly to any particular inflaton or cyclic scalar field potential. In fact, it does not assume that either scenario is driven by a scalar field at all. But, one might ask, how does our condition on the equation of state translate into a condition on an inflaton potential? The answer is simple: it means that the potential is characterized by a single dimensionful scale, typically H_I, the Hubble parameter during inflation. For example, for many models the effective potential is well characterized as $M^4 f(\phi/M)$, where ϕ is the inflaton field, $H_I \approx M^2/M_{\rm Pl}$, where $M_{\rm Pl}$ is the Planck mass, and $f(x)$ is a smooth function which, when expanded in ϕ/M, has all dimensionless parameters of the same order [14]. In these cases, to produce inflationary models in which there are rapid changes in the equation of state in the last $\mathcal{N}$-folds, sharp features have to be introduced in the inflaton potential: bumps, wiggles, steep waterfalls, etc.

But recall that the inflaton field is rolling very slowly throughout inflation, including the last $\mathcal{N}$ e-folds. Typically, ϕ rolls a short distance, $\Delta\phi \ll M$, during the last $\mathcal{N}$ e-folds. Hence, any sharp features must take place over a range $\delta\phi \ll \Delta\phi \ll M$, or equivalently, by introducing new fields or new mass scales much greater than M in the inflaton potential. For the purposes of comparing the inflationary and or cyclic predictions, it makes most sense to consider the class with fewest parameters and simplest uniform behavior of the equation of state, a class which is also well motivated in both models.

Recently, Gratton *et al.* [13] analyzed the conditions on the equation of state w required in order for quantum fluctuations in a single scalar field to produce nearly scale-invariant density perturbations, including models which (in the four dimensional effective description) bounce from a contracting to an expanding phase. Their analysis showed that there are only two cases which avoid extreme fine-tuning of initial conditions and/or the effective potential: $w \approx -1$ (inflation) and $w \gg 1$ (the ekpyrotic or cyclic scenario).

Following Gratton *et al.* [13], we discuss the production of long wavelength perturbations in the gauge invariant Newtonian potential Φ, which completely characterizes the density perturbation. Defining $u \equiv a\Phi/\phi'$ (henceforth, primes denote differentiation with respect to conformal time τ), then a Fourier mode of u with wave number k, u_k, obeys the differential equation

$$u_k'' + \left(k^2 - \frac{\beta(\tau)}{\tau^2}\right)u_k = 0, \tag{1}$$

with

$$\beta(\tau) \equiv \tau^2 H^2 a^2\left\{\bar{\epsilon} - \frac{(1-\bar{\epsilon}^2)}{2}\left(\frac{d\ln\bar{\epsilon}}{d\mathcal{N}}\right) + \frac{(1-\bar{\epsilon}^2)}{4}\left(\frac{d\ln\bar{\epsilon}}{d\mathcal{N}}\right)^2 - \frac{(1-\bar{\epsilon})^2}{2}\frac{d^2\ln\bar{\epsilon}}{d\mathcal{N}^2}\right\}, \tag{2}$$

where $H = a'/a^2$ is the Hubble parameter, and where $\bar{\epsilon}$ is related to the equation of state parameter w by

$$\bar{\epsilon} \equiv \frac{3}{2}(1 + w). \tag{3}$$

We have introduced the dimensionless time variable $\mathcal{N}$, defined by

$$\mathcal{N} \equiv \ln\left(\frac{a_{\rm end}H_{\rm end}}{aH}\right), \tag{4}$$

where the subscript "end" denotes that the quantity is to be evaluated at the end of the inflationary expansion phase or ekpyrotic contraction phase (corresponding to $w \gg 1$). Note that $\mathcal{N}$ measures the number of e-folds of modes which exit the horizon before the end of the inflationary or ekpyrotic phase. [N.B. $d\mathcal{N} = (\bar{\epsilon} - 1)dN$, where $N = \ln a$ in Ref. [13].] Indeed, defining as usual the moment of horizon crossing as $k_{\mathcal{N}} = aH$ for a given Fourier mode with comoving wave number $k_{\mathcal{N}}$, then

$$\mathcal{N} = \ln\left(\frac{k_{\rm end}}{k_{\mathcal{N}}}\right), \tag{5}$$

where $k_{\rm end}$ is the last mode to be generated.

For nearly constant w (or constant $\bar{\epsilon}$), the unperturbed equations of motion have the approximate solution

$$a(\tau) \sim (-\tau)^{1/(\bar{\epsilon}-1)}, \qquad H = \frac{1}{(\bar{\epsilon}-1)a\tau}. \tag{6}$$

Substituting the second of these expressions in β, we find

$$\beta(\tau) \approx \frac{1}{(1-\bar{\epsilon})^2}\left\{\bar{\epsilon} - \frac{(1-\bar{\epsilon}^2)}{2}\left(\frac{d\ln\bar{\epsilon}}{d\mathcal{N}}\right)\right\}, \tag{7}$$

where we have assumed that the higher-order derivative terms $d^2\ln\bar{\epsilon}/d\mathcal{N}^2$ and $(d\ln\bar{\epsilon}/d\mathcal{N})^2$ are much smaller than $d\ln\bar{\epsilon}/d\mathcal{N}$.

With the approximation that β is nearly constant for all modes of interest, Eq. (1) can be solved analytically, and the resulting deviation from scale invariance is simply given by the master equation

$$n_s - 1 \approx -2\beta \approx -\frac{2}{(1-\bar{\epsilon})^2}\left\{\bar{\epsilon} - \frac{(1-\bar{\epsilon}^2)}{2}\left(\frac{d\ln\bar{\epsilon}}{d\mathcal{N}}\right)\right\}. \tag{8}$$

Inflation.—Inflation is characterized by a period of superluminal expansion during which $w \approx -1$; that is, $\bar{\epsilon} \ll 1$. In this case, Eq. (8) reduces to

$$n_s - 1 \approx -2\bar{\epsilon} + \frac{d\ln\bar{\epsilon}}{d\mathcal{N}}, \tag{9}$$

as derived by Wang *et al.* [14].

The next step consists in rewriting the above in terms of $\mathcal{N}$ only. For this purpose, we need a relation between $\bar{\epsilon}$ and $\mathcal{N}$. During inflation, the Hubble parameter is nearly constant, but the end means that H begins to change significantly. So, if we are considering the last $\mathcal{N}$ e-folds, then, using Eqs. (6) and the definition of $\mathcal{N}$ [see Eq. (4)], it must be that H decays by a factor of order unity over those $\mathcal{N}$ e-folds or

$$\frac{H_{\rm end}}{H} = \left(\frac{a}{a_{\rm end}}\right)^{\bar{\epsilon}} \approx e^{-\bar{\epsilon}\mathcal{N}} \approx e^{-1}, \tag{10}$$

or

$$\bar{\epsilon} \approx \frac{1}{\mathcal{N}}. \tag{11}$$

Assuming that this relation holds approximately for all relevant modes, we may substitute in Eq. (9) and obtain

$$(n_s - 1)_{\rm inf} \approx -\frac{2}{\mathcal{N}} - \frac{1}{\mathcal{N}} = -\frac{3}{\mathcal{N}}. \tag{12}$$

Note that, in this approximation, the two terms on the right-hand side of Eq. (12) are both of order $1/\mathcal{N}$. Figuring that our approximation is good to order $1/\mathcal{N}$ or a few percent, the result is in agreement with the tilt predicted by simple inflationary models [15].

To obtain a numerical estimate of n_s, we may derive an approximate value for $\mathcal{N}$ from the observational constraint that the amplitude of the density perturbations, $\delta\rho/\rho$, be of order 10^{-5}. In the simplest inflationary models, $\delta\rho/\rho$ is given by [6]

$$\frac{\delta\rho}{\rho} \approx \left(\frac{T_r}{M_{\rm Pl}}\right)^2 \bar{\epsilon}^{-1/2} \approx \left(\frac{T_r}{M_{\rm Pl}}\right)^2 \mathcal{N}^{1/2} \sim 10^{-5}, \tag{13}$$

where T_r is the reheat temperature. On the scale of the observable Universe today, $\mathcal{N}$ has the value [see Eq. (4)]

$$\mathcal{N} = \ln\left(\frac{a_{\rm end}H_{\rm end}}{a_0 H_0}\right) \approx \ln\left(\frac{T_r}{T_0}\right), \tag{14}$$

where a_0, H_0, and T_0 are, respectively, the current values of the scale factor, Hubble parameter, and (photon) temperature. For simplicity, we have assumed that $H_{\rm end} \sim T_r^2/M_{\rm Pl}$. Combining Eqs. (13) and (14), we obtain the constraint

$$e^{\mathcal{N}}\mathcal{N}^{1/4} \approx 10^{-5/2}\frac{M_{\rm Pl}}{T_0}, \tag{15}$$

which implies $\mathcal{N} \approx 60$. It follows that $T_r \sim 10^{16}$ GeV.

If we substitute $\mathcal{N} \approx 60$ in Eq. (12), we obtain $n_s \approx 0.95$, within a percent or two of what is found for the simplest slow-roll and chaotic potentials [16,17].

The prediction for the ratio of tensor to scalar contributions to the quadrupole of the CMB for a model with 70% dark energy and 30% matter is, then, [16–18]

$$T/S \approx 13.8\bar{\epsilon} \approx \frac{13.8}{\mathcal{N}} \approx 23\%, \tag{16}$$

which is very pleasing because it is in the range which is potentially detectable in the fluctuation spectrum and/or the CMB polarization in the near future [19]. [The WMAP Collaboration [20] uses a different convention for T/S, defining $(T/S)_{\rm WMAP}$ as the ratio of tensor to scalar amplitude of the primordial spectrum. The conversion factor to our T/S is $(T/S)_{\rm WMAP} \approx 1.16\,(T/S)$.]

It is sometimes said that it is easy to construct models where T/S is very small, less than 1%, say. The argument is that the amplitude of tensor fluctuations is proportional to H^2, and a modest decrease in the energy scale for inflation reduces the tensor amplitude significantly. However, one must also consider Eq. (16) combined with Eq. (11). From Eq. (16), making T/S less than 1%, for instance, requires $\bar{\epsilon} < 10^{-3}$, which implies $\mathcal{N} > 1000$. Since we are interested in T/S at $\mathcal{N} \approx 60$, however, the only way to accommodate such a small $\bar{\epsilon}$ at $\mathcal{N} \approx 60$ is to have $\bar{\epsilon}$ make a rapid change at some point between $\mathcal{N} \approx 60$ and the end of inflation. This is precisely what is done in models which yield a small T/S ratio. [Restated in terms of the inflation potential $V(\phi)$, in order to have $T/S \approx 13.8\bar{\epsilon} \approx 28\,(d\ln V/d\phi)^2 \ll 1\%$, it must be that $d\ln V/d\phi \ll 0.02$, which is too small if inflation is to end in 60 e-folds unless one introduces a very rapid change in the slope during the last 60 e-folds.]

Ekpyrotic or cyclic models.—The ekpyrotic phase is characterized by a period of slow contraction with $w \gg 1$; that is, $\bar{\epsilon} \gg 1$. In this case, Eq. (8) reduces to [13]

$$n_s - 1 \approx -\frac{2}{\bar{\epsilon}} - \frac{d\ln\bar{\epsilon}}{d\mathcal{N}}. \tag{17}$$

Notice that this relation can be transformed into the expression for inflation, Eq. (9), by replacing $\bar{\epsilon} \to 1/\bar{\epsilon}$. Note further that, for all cosmologies, the scale factor is $a \propto t^{1/\bar{\epsilon}} \propto H^{-1/\bar{\epsilon}}$, where t is proper time. Hence, inflation ($\bar{\epsilon} \ll 1$) has a rapidly varying and H nearly constant, whereas the ekpyrotic or cyclic model ($\bar{\epsilon} \gg 1$) has H varying and a nearly constant. This suggests an interesting duality between the inflationary and ekpyrotic or cyclic models that reflects itself in the final results.

If the scale factor $a(\tau)$ is nearly constant during the ekpyrotic (contraction) phase, then the phase ends when $a(\tau)$ begins to change significantly. In particular, the condition that the scale factor $a(\tau)$ decays by a factor of order unity during the last $\mathcal{N}$ e-folds reads

$$\frac{a_{\rm end}}{a} = \left(\frac{aH}{a_{\rm end}H_{\rm end}}\right)^{1/(\bar{\epsilon}-1)} \approx e^{-\mathcal{N}/\bar{\epsilon}} \approx e^{-1} \tag{18}$$

[the analog of Eq. (10) for inflation], which implies

$$\bar{\epsilon} \approx \mathcal{N} \tag{19}$$

[to be compared with Eq. (11) for inflation]. Substituting this expression into Eq. (17), one obtains

$$(n_s - 1)_{ek} \approx -\frac{2}{\mathcal{N}} - \frac{1}{\mathcal{N}} = -\frac{3}{\mathcal{N}}. \tag{20}$$

This is the key relation for the ekpyrotic or cyclic models.

In the inflationary case, we estimated $\mathcal{N}$ by using the constraint on the amplitude of density perturbations, $\delta\rho/\rho \sim 10^{-5}$. For ekpyrotic and cyclic models, this constraint involves more parameters and is therefore not sufficient by itself to fix $\mathcal{N}$ [12,21]. To estimate $\mathcal{N}$, we rewrite Eq. (4) as

$$\mathcal{N} \approx \ln\left(\frac{T_r}{T_0}\right) + \ln\left(\frac{a_{\rm end}H_{\rm end}}{a_r H_r}\right), \tag{21}$$

where the subscript r denotes the onset of the radiation-dominated phase. In inflation, we have $a_{\rm end} \approx a_r$ and $H_{\rm end} \approx H_r$. In the ekpyrotic or cyclic model, however, the end of ekpyrosis occurs during the contracting phase whereas the onset of radiation domination is during the expanding phase. To estimate the ratio $a_{\rm end}H_{\rm end}/a_r H_r$, we note that, from approximately the end of ekpyrosis, through the bounce, and up to the onset of radiation domination, the Universe is dominated by scalar field kinetic energy; i.e., $w \approx 1$ [4,5]. From Eqs. (3) and (6), we find $a \approx (-\tau)^{1/2} \sim H^{-1/3}$, and therefore

$$\frac{a_{\rm end}H_{\rm end}}{a_r H_r} \approx \left(\frac{H_{\rm end}M_{\rm Pl}}{T_r^2}\right)^{2/3}. \tag{22}$$

Substituting in Eq. (21), we find

$$e^{\mathcal{N}} = \left(\frac{H_{\rm end}^2}{T_r M_{\rm Pl}}\right)^{1/3}\frac{M_{\rm Pl}}{T_0}, \tag{23}$$

which is the analog of Eq. (15).

The constraints on $H_{\rm end}$ and T_r in cyclic models are analyzed in Ref. [21] and the range of allowed values is presented. Central values are $T_r \approx 10^5$ GeV and $H_{\rm end} \approx 10^5$ GeV, which, from Eq. (23), implies $\mathcal{N} \approx 60$. (By pushing parameters, $\mathcal{N}$ can be made to vary 20% or so one way or the other.) Substituting $\mathcal{N} = 60$ in the expression for the tilt gives $n_s \approx 0.95$, the same estimate obtained for inflation.

Conclusions.—Remarkably, our estimates for the typical tilt in the inflationary and ekpyrotic or cyclic models are virtually identical. Both models predict a red spectrum, with spectral slope

$$n_s - 1 \approx -\frac{3}{\mathcal{N}}. \tag{24}$$

Furthermore, when adding observational constraints such as the Cosmic Background Explorer (COBE) constraint that the amplitude of density fluctuations be of order 10^{-5}, both models yield $\mathcal{N} \approx 60$. This results in an identical prediction for the spectral tilt of $n_s \approx 0.95$. Furthermore, in both models, the time variation of the equation of state contributes a correction of $\mathcal{O}(1)$ that reddens the spectrum. We have seen that this occurs because there is fascinating duality ($\bar{\epsilon} \rightarrow 1/\bar{\epsilon}$) between inflationary and ekpyrotic or cyclic conditions. This result was neither planned nor anticipated and suggests a deep connection between the expanding inflationary phase and the contracting ekpyrotic or cyclic phase. The key difference is that inflation also predicts a nearly scale-invariant spectrum of gravitational waves with a detectable amplitude. The predicted ratio of tensor to scalar CMB multipole moments at low l is $T/S \approx 20\%$. The tensor spectrum from cyclic models is strongly blue and exponentially small on cosmic scales [3,22].

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