Mechanism for Spiral Wave Breakup in Excitable and Oscillatory Media

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We study spiral wave breakup using a Fitzhugh-Nagumo–type system. We find that spiral wave breakup can occur near the core or far from it in both excitable and oscillatory regimes. There is a faraway breakup scenario in both excitable and oscillatory media that depends on long wavelength modulation modes. We observed three distinct scenarios, including one that involves breakup that does not develop into turbulence. However, we find that the mechanisms behind these three scenarios are the same: they are caused by the interaction between the dispersion relation and the asymptotic behavior of the modulation mode. The difference in phenomenology is due to the asymptotic behavior of the modulation mode.

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The transition from a regular pattern to spatiotemporal chaos in extended systems remains a challenging problem in nonlinear dynamics [1]. In particular, spiral wave instability in reaction-diffusion systems is a robust phenomenon observed in both experiments and numerical simulations [2–9]. Phenomenologically, two different breakup scenarios have been documented in experiments and numerical simulations: Doppler instability [3,4] and the convective (absolute) Eckhaus instability [5–8]. The first instability occurs in excitable media, and the spiral waves break up near the spiral core. When the stable spiral wave becomes unstable via Hopf bifurcation, the spiral tip meanders, and the Doppler effect becomes so large that adjacent waves interact and break. This scenario provides a possible mechanism for cardiac fibrillation [4].

In the second instability, which appears in oscillatory media, spiral wave breakup happens far from the spiral core, and the region near the core remains unchanged, while the wave propagating outwards from the spiral core becomes subject to a longitudinal instability (Eckhaus instability). This scenario has been observed in experiments on the Belousov-Zhabotinsky (BZ) reaction [5]; the authors attribute it to the presence of a convective Eckhaus instability. Recently, this scenario was explained by the appearance of a global mode that asymptotes to the absolute Eckhaus instability in large systems [6]. It has been suggested that near-core breakup is characteristic of excitable media, while breakup far from the core should only happen in excitable media [7].

In this Letter, we investigate the break up of spiral waves leading to defect-mediated turbulence. To do that, a 2D Fitzhugh-Nagumo–type model is considered [9]. The model consists of the following equations:

$$\frac{\partial u}{\partial t} = -\frac{1}{\varepsilon}u(u-1)\left(u-\frac{b+v}{a}\right)+\nabla^2 u,$$

$$\frac{\partial v}{\partial t} = f(u) - v,$$

$$f(u) = \begin{cases} 0, & 0 \le u < 1/3, \\ 1-6.75u(u-1)^2, & 1/3 \le u < 1, \\ 1, & 1 < u, \end{cases}$$

with no-flux boundary conditions. In simulations, we use Euler integration with a space step $dx = dy = 0.45$ and a time step $dt = 0.03$. Similar results were obtained with smaller space and time steps. Generally, we take the parameter a in the model to be constant ($a = 0.84$) and we change b from negative to positive, changing the corresponding local dynamics from oscillatory (if $b < 0$) to excitable (if $b > 0$).

To study the spiral wave dynamics, we record the tip trajectory, then measure the outer diameter (R) of the tip orbit (defined by the isoline $u = 0.3$) when the spiral wave behavior is stable or meandering. When $b = -0.045$ [Fig. 1(a)], it is well known that the system first encounters a Hopf bifurcation around $\varepsilon = 0.045$. Before that, the spiral wave is stable; beyond that, the spiral wave meanders. Consistent with the Hopf bifurcation, note that the R curve follows an approximate square root relation for ε from 0.45 to 0.553. After that, the dynamics changes dramatically, and spiral wave breakup occurs for $\varepsilon \in (0.553, 0.056)$. Although the usual view is that spiral wave breakup occurs far from the core in oscillatory media, here the spiral wave breaks up near the core region [Fig. 1(b)]. Further increasing ε, there is a transition back to a meandering spiral wave. In this regime, R is much larger than before breakup but still much smaller than the size of the system. As a result, we can exclude the possibility that the spiral wave breakup in $e \in (0.553, 0.056)$ is caused by the collisions between the core and the boundary. Another spiral wave breakup regime occurs when $\varepsilon > 0.77$. As opposed to the previous

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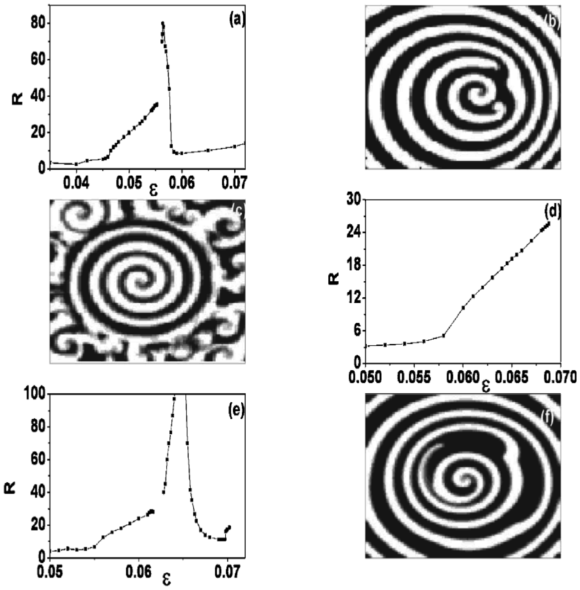

FIG. 1. Three scenarios of spiral wave breakup. The size of the system is 200×200 for the snapshots (a) The diameter R of the orbit of the spiral tip as a function of ε at $b = -0.045$; (b) snapshot of spatial pattern at $b = -0.045, \varepsilon = 0.0555$, near-core breakup of spiral is shown; (c) snapshot at $b = -0.045, \varepsilon = 0.078$, faraway breakup of spiral wave is shown; (d) R vs ε at $b = 0.05$; (e) R vs ε at $b = 0.001$ [10]; (f) novel breakup scenario is shown, for $b = 0.001$ and $\varepsilon = 0.0705$.

breakup regime, here the spiral wave breaks up far away from the core. [Fig. 1(c)].

For the excitable regime, we take $b = 0.05$, and again study R as a function of ε [Fig. 1(d)]. As opposed to the oscillatory regime [Fig. 1(a)], here only one transition to breakup is observed. The spiral wave breaks up near the core, as in Fig. 1(b). What does the dynamics look like when we take the value of b between -0.045 and 0.05? To explore this, we take $b = 0.001$ (still in the excitable regime). Figure 1(e) shows the corresponding R curve. In Fig. 1(e) there are still two breakup regimes, but the first breakup regime where the spiral wave breaks up near the core has shifted markedly toward the second breakup regime. If we further increase b towards 0.05, we find that the first breakup region finally merges with the second region, just as in Fig. 1(d). The second breakup region is much more interesting. According to the usual view, the spiral wave should break up near the core because it is in the excitable regime. However, note in Fig. 1(e) that this second breakup regime occurs at large ε, corresponding to the faraway breakup regime in Fig. 1(a). To resolve this paradox, we plot a snapshot where the first breakup begins in Fig. 1(f). Note that the spiral wave breaks up several wavelengths away from the core, but the breakup does not develop into a chaotic sea as in Fig. 1(c) or to defect-mediated turbulence as in Fig. 1(b). Instead a novel pattern is formed, in which the spatial pattern is divided into two parts: inside is a rotating spiral wavelet with tail, and outside is a distorted target wave. The tail of the inside wave collides with the back of the outer wave from time to time; the new breakup creates a new tail and a new target wave front. Further increasing ε beyond a critical value, the outside distorted target wave yields to the chaotic sea. As opposed to the case in Fig. 1(c), the coherent spiral wavelet cannot sustain itself forever. It will be replaced by other spiral wavelets grown out of the chaotic sea.

Phenomenologically, the three scenarios of spiral wave breakup are quite different, but the mechanisms behind them are the same. In the faraway breakup scenario, a global mode was proposed and validated by experiment [8], but is there a global mode in the near-core breakup scenario? To investigate this problem, we study a system whose size is 400×400. We calculated the power spectra for different sites when $b = 0.05$ and $\varepsilon = 0.0684$ [Figs. 2(a)–2(c)]. The core locates around (200, 200). Note that there are several discrete peaks in Fig. 2(a). After some simple manipulations, we find that there are only two fundamental frequencies $f_1 = 0.199$ and $f_2 = 0.0238$. The other peaks are linear combinations of f_1 and f_2. The primary frequency f_1 is the frequency of the stable spiral wave. f_2 is the secondary frequency caused by the Eckhaus instability which modulates the spiral wave. Similar primary and secondary frequencies are seen independently of location in Figs. 2(b) and 2(c); hence, these plots verify the existence of a global mode after the stable spiral wave loses its stability. These power spectra differ in the amplitudes of the various modes: the peak of the second frequency near the core [Fig. 2(a)] is much higher than that far away from the core [Fig. 2(c)]. The power spectrum also shows

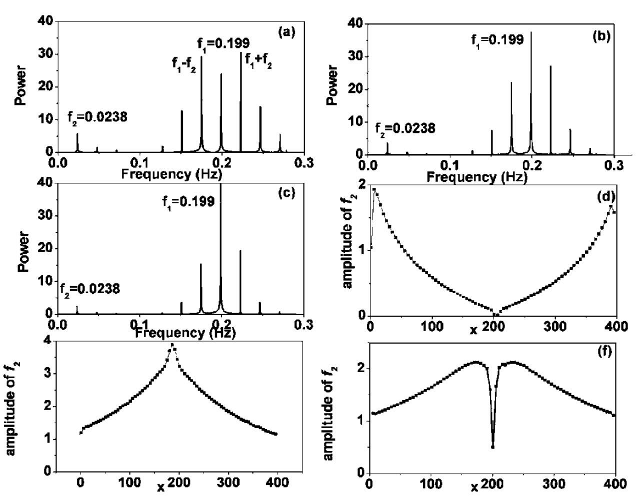

FIG. 2. Power spectra are shown for $b = 0.05, \varepsilon = 0.0684$ for different locations (a) site (200, 210); (b) site (200, 300); (c) site (200, 390); (d)–(f) strength of the Eckhaus mode versus the spatial location along a line which crosses the core region: (d) $b = -0.045, \varepsilon = 0.077$; (e) $b = 0.05, \varepsilon = 0.0684$; (f) $b = 0.001, \varepsilon = 0.0698$.

that the Eckhaus instability which generates the modulation mode of the spiral wave is absolute not convective. This is because the continued presence of a modulation mode, causing the second frequency, implies that the mode is *not* convecting.

However, the existence of the second mode caused by the Eckhaus instability does not guarantee the spiral wave breakup; it just causes the spiral wave to meander. To investigate the cause of breakup, we plot the amplitude of f_2 along a line that crosses the core region for different parameter values [Figs. 2(d)–2(f)]. Just prior to the faraway breakup transition, the amplitude of f_2 increases with the distance from the core [Fig. 2(d)]. By contrast, in the near-core breakup scenario, the opposite is seen: f_2 decreases with the distance from the core [Fig. 2(e)]. And in the new, third breakup scenario, the amplitude of f_2 first increases, then decreases. The change in the amplitude of f_2 indicates that the modulations of the spiral wave by the secondary mode are different at different spatial locations. To see the consequence of this modulation, we trace the change of the temporal period along the same lines as in Figs. 2(d)–2(f). Here we define the period as the time interval between successive maxima of the variable u. The system was run for several thousand periods after a transient. The maximum and the minimum of the period ($p_{\max}$ and $p_{\min}$) at all sites along the line are shown in Fig. 3. The parameters are chosen in the meandering spiral wave regime, but close to the breakup regime. Three distinct cases were found. (1) The faraway breakup case: here, the maximum (or minimum) of period increases (or decreases) with the distance away from the core outside of the core area [Fig. 3(a)]. (2) The near-core breakup case: here, contrary to the faraway breakup case, $p_{\max}$ (or $p_{\min}$) reaches its maximum (or minimum) around the spiral wave core (the size of the core $R = 25$ [Figs. 1(d) and 3(b)]. (3) A novel breakup pattern: this is similar to the near-core breakup case, *but* the location where the maximum (or minimum) of $p_{\max}$ (or $p_{\min}$) reaches its maximum (or minimum) is far away from the core [$R = 18$ in Fig. 1(e)].

It is important to note that the period of the propagating wave cannot take an arbitrary value, due to the dispersion relation. In excitable media, it is well known that there exists a minimum period due to the refractory part of the local dynamics, below which conduction block leads to failure of wave propagation [4]. However, to our knowledge, what the dispersion relation is in oscillatory media is an unknown problem. Here we use three protocols to study the dispersion relation: (1) wave propagation in a 1D version ($dx = 0.39, dt = 0.015$, the size of the system is 1600); (2) planar waves propagating through a 2D square region; (3) target waves in a 2D system with no-flux boundary conditions. The first two protocols do not consider the effect of curvature and the rightmost boundary obeys a no-flux boundary condition. We drive the leftmost site(s) with sinusoidal forcing. In the third case, the effect of curvature is taken into consideration, and we drive the center of the system with sinusoidal forcing. After the transient, a steady planar (or target) wave could be established downstream for a certain range of driving frequencies. The wave number can be obtained by spatial Fourier analysis. The results are shown in Fig. 4. When the driving frequency becomes too large or too small, the planar (or target) wave cannot propagate at the period of the driving force. Actually, there exist two critical driving frequencies; beyond either of them, the 1D system goes to chaotic behavior.

Now, based on the results in Figs. 3 and 4, we can explain why the spiral wave breaks up in different ways. In the near-core breakup case, the modulation mode of the spiral wave reaches its maximum amplitude around

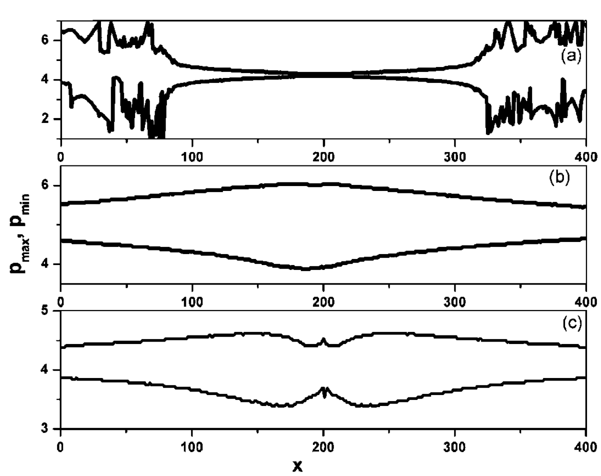

FIG. 3. The maximum period and the minimum period measured as a function of location. (a) $b = -0.045, \varepsilon = 0.078$; (b) $b = 0.05, \varepsilon = 0.0684$; (c) $b = 0.001, \varepsilon = 0.0698$.

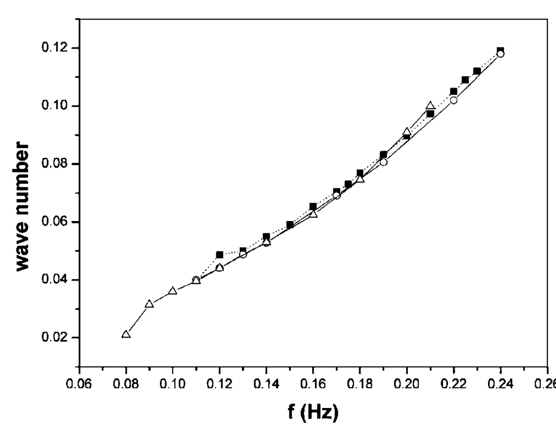

FIG. 4. The dispersion relation for $b = -0.045, e = 0.07$. Open circles are for planar waves in a 2D system, solid squares are for a 1D system, and open triangles are for a target wave. There is no difference between the planar waves in 1D and 2D systems. The effect of curvature on the target waves is just to shift the curve down to lower frequencies.

the core. Correspondingly, $p_{\max}$ (or $p_{\min}$) reaches its maximum (or minimum). The increase of ε leads to the increase of $p_{\max}$ and decrease of $p_{\min}$. In the excitable system, the decrease of $p_{\min}$ with ε eventually leads $p_{\min}$ to become smaller than the minimum period permitted by the dispersion relation. In the oscillatory medium, generally both the increase of $p_{\max}$ and the decrease of $p_{\min}$ with ε at a given b could violate the dispersion relation and lead to spiral wave breakup. In the model studied here, the violation of $p_{\min}$ leads to the breakup. In the near-core case, once the breakup occurs around the core region, the interaction between the defects due to the breakup and the core is strong enough to make the defects move away, and more defects will be generated around them, leading finally to defect-mediated turbulence. In the novel case, breakup begins when the dispersion relation is violated. The first breakup occurs only in a thin annulus because the dispersion relation still holds outside the annulus on both sides. Because the breakup occurs far away from the core, the interaction between the core and the defect is negligible. As a result, we see a breakup but no spatiotemporal chaos. Only when the width of the annulus becomes large enough that more defects can be generated in the annulus can we see a chaotic sea due to the interaction between the defects surrounding the short-lived spiral wavelet. Because of the outward propagation of the wave arm emitted by the core, here the downstream chaotic sea is caused by convection. In the faraway breakup case, the modulation mode becomes stronger with the distance away from the core. This results in different phenomena in different system sizes: we can see no breakup for small systems, while a chaotic sea surrounding the laminar state is observed for large systems [see Fig. 3(a)]. In this case, once the dispersion relation is violated at a certain distance from the core, the propagation downstream must violate the dispersion relation, leading finally to the chaotic sea. In this sense, it is different from the chaotic sea surrounding the laminar state in the novel case; this faraway scenario yields an absolute chaotic sea.

The modulation mode caused by the Eckhaus instability is of the long wavelength type. The snapshot of the 2D system is shown in Fig. 5(a). The period for each site during a small time section $(t, t + p_{\max})$ is recorded in Fig. 5(b). The variation of the period in 2D space shows another spiral wave that signifies the spatial character of the modulation mode. Compared with Fig. 5(a), the long wavelength in Fig. 5(b) is obvious. Actually, the pattern shown in Fig. 5(b) is the so-called overspiral addressed in some experiments [4].

In conclusion, we investigated an Fitzhugh-Nagumo–type system that can behave like an excitable or an oscillatory system by the proper choice of parameters. We find that whether the spiral wave undergoes faraway breakup or near-core breakup cannot be decided by knowing whether the system is in its excitable or oscillatory regimes. Both scenarios can occur in either type medium. We find that the long wavelength modulation mode is sustained in both excitable and oscillatory media. Phenomenologically, we found that there is a third scenario of spiral wave breakup, in addition to near-core breakup and faraway breakup, which consists of a broken spiral wave surrounded by a target pattern. However, we found that the mechanisms supporting these three scenarios are the same; they are caused by the interaction between the dispersion relation and the asymptotic behavior of the modulation mode. The difference in phenomenology is caused by the specific asymptotic behavior of the modulation mode: whenever the modulation mode causes the spiral wave to violate the dispersion relation, wave breakup occurs.

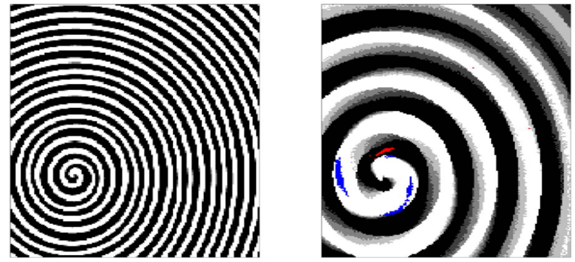

FIG. 5 (color online). Superspiral wave is shown for $b = -0.02$, $\varepsilon = 0.07$. (a) The wave pattern of the original system, (b) the variation of the period. The spatial character of the modulation mode is clearly observed.

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[1] M. C. Cross and P. C. Hohenberg, Rev. Mod. Phys. **65**, 851 (1993).
[2] Z. Qu, F. Xie, A. Garfinkel, and J. N. Weiss, Ann. Biomed. Eng. **28**, 755 (2000).
[3] M. Bar, M. Hildebrand, M. Eiswirth, M. Falcke, H. Engel, and M. Neufeld, Chaos **4**, 499 (1994).
[4] Q. Ouyang, H. L. Swinney, and G. Li, Phys. Rev. Lett. **84**, 1047 (2000).
[5] Q. Ouyang and J. M. Flesselles, Nature (London) **379**, 143 (1996).
[6] S. M. Tobias and E. Knobloch, Phys. Rev. Lett. **80**, 4811 (1998).
[7] M. Bar and M. Or-Guil, Phys. Rev. Lett. **82**, 1160 (1999).
[8] L. Zhou and Q. Ouyang, Phys. Rev. Lett. **85**, 1650 (2000).
[9] M. Bar and M. Eiswirth, Phys. Rev. E **48**, R1635 (1993).
[10] A transition from the meandering spiral wave with outward petal to the one with inward petal occurs around $e = 0.065$.