Granular Elasticity without the Coulomb Condition

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A self-contained elastic theory is derived which accounts both for mechanical yield and shear-induced volume dilatancy. Its two essential ingredients are thermodynamic instability and the dependence of the elastic moduli on compression.

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Stable sandpiles show clear elastic behavior, though conventional elasticity theory cannot be appropriate, as sandpiles also possess the steepest slope. The associated angle, the Coulomb angle φ, has a typical value of around 30° for dry sand. So granular materials may be taken to interpolate between fluid (no elasticity, $\varphi = 0°$) and solid (conventional elasticity, $\varphi \geq 90°$). This is usually understood in terms of the solid friction law: Subject to gravitational pull, a sand grain resting on a slope of angle θ experiences the force $N = \rho G \cos\theta$ normal to the surface, and $S = \rho G \sin\theta$ along it, with $S/N = \tan\theta$ (ρ is the mass density and G the gravitational constant). Since static friction will only sustain a maximal S/N, there is a maximal θ, which one may identify with the Coulomb angle φ, or $S/N \leq \tan\varphi$. Now, requiring this to hold both along any the plane and everywhere in the bulk, one may reinterpret S and N as components of the stress tensor σ_{ij}, respectively, tangential and normal to the given plane. Then the inequality may, with σ_1 and σ_3 as the largest and smallest eigenvalues of σ_{ij}, be written as

$$|(\sigma_1 - \sigma_3)/(\sigma_1 + \sigma_3)| \leq \sin\varphi. \quad (1)$$

This is the "Coulomb yield condition" or "Coulomb law of internal friction," a textbook formula of soil mechanics [1], employed to impose mechanical yield upon conventional elasticity. Although this formula captures essential granular physics, it possesses a number of deficiencies. First, the Coulomb condition preempts what should have been the result of a proper theory: Ideally, one would like to start from a continuum theory, calculate the spatial dependence of the stress with appropriate boundary conditions, and arrive—with some substantiated understanding—at the fact that there is a maximal shear stress in sand, above which the system is mechanically unstable. This includes especially an expression for the Coulomb angle φ. The Coulomb law postulates the last bit and employs it backward.

Second, the Coulomb condition takes for granted that mechanical yield is determined by a unique φ, independent of geometry. Third, it is not obvious that granular materials behave as conventional solids up to the yield point, without any precursor behavior. Fourth, the Coulomb condition partially contradicts conventional elasticity, and our understanding is rendered regrettably precarious. Last, but not least, Reynolds dilatancy [2]—the volume expansion concurring with shear motion in granular materials—should be an integral part of mechanical yield, yet it is usually ignored by the Coulomb law [3]. Imagine a pile of stacked steel balls, and envisage how a shearing displacement lifts the balls from their close-packed positions and gives rise to volume expansion—which eventually leads to yield. In experiments [4] and simulations [5] this is what has been observed.

Although the Coulomb condition appears unique to granular systems, its sole purpose is to account for mechanical yield, or the lack of elastic configurations for certain values of stress and strain. Yield is a widespread phenomenon in many solids at high stresses, which are well accounted for by linear elasticity at lower values of stress. "Low, high" are of course relative concepts, and the noteworthy point is that being characterized by a quadratic elastic free energy and a linear stress,

$$f_{el} = \tfrac{1}{2} K_b u_{nn}^2 + K_a u_{k\ell}^0 u_{k\ell}^0, \quad (2)$$

$$\sigma_{ij} = -K_b u_{nn} \delta_{ij} - 2K_a u_{ij}^0, \quad (3)$$

linear elastic theory cannot possibly account for yield, as it provides stable elastic solutions for arbitrary strains and stresses, however high. (u_{nn} is the trace of the strain tensor u_{ij}, $u_{kl}^0 \equiv u_{kl} - u_{nn}\delta_{kl}/3$ is its traceless part, and K_b, K_a are the constant compressional and shear moduli.) Yet contrary to prevalent perception, this is not a general feature of elasticity: Adding nonlinear terms to Eqs. (2) and (3) may well render given elastic solutions unstable for some variable range. And it appears obvious that elastic instabilities, or more generally the lack of elastic solutions, are to be identified with yield, the lack of elastic configurations. Doing so embraces yield as the generic phenomenon that it is, and does away with extraneous inputs such as the Coulomb condition. Note also that yield therefore marks the end of the range of validity for elasticity. Only a more comprehensive theory including dissipative terms is able to describe what then happens, typically plastic flows.

In granular materials, one need not look far for terms that fit the above description. Consider two solid spheres in contact [6] to find $U \sim h^{5/2}$, $f \sim h^{3/2}$, where U is the elastic energy, f the applied force, and h the relative change in height. The latter relation is not linear, because the area of contact between the two spheres and the amount of compressed mass increase with h. Assuming one can scale up this two-body result to granular materials, in usual parlance, that they possess "Hertz contacts," we identify U, f, h, respectively, with f_{el}, σ_{nn}, u_{nn} of Eqs. (2) and (3), and conclude $K_b \sim u_{nn}^{1/2} \sim \sigma_{nn}^{1/3}$ is no longer constant. Realistically, with more than two grains in contact, K_b will still depend on σ_{nn} and vanish with it, so one may more generally take $K_b \sim \sigma_{nn}^{\beta}$. And since the physics of increasing contact area is similar for shear, also take $K_a \sim \sigma_{nn}^{\beta}$. Evesque and de Gennes employed these elastic moduli in the stress of Eq. (3) to successfully render the pressure saturating in silos [7].

Aiming to generalize this quasielastic theory and embed it into a consistent thermodynamic framework, we made the following observation: With $K_b, K_a \sim u_{nn}^a$ and the free energy f_{el} retaining its form of Eq. (2), the stress σ_{ij}—given by general considerations essentially as $\partial f_{el}/\partial u_{ij}$—is necessarily modified; Eq. (3) is clearly correct only if K_a, K_b are constant. Our serendipitous finding reported below is that the additional terms of σ_{ij} suffice to account for mechanical yield and volume dilatancy by rendering elastic solutions unstable in a range of parameters appropriate for granular materials. (Assuming as above that it is the stress which retains its form, no f_{el} exists such that $\sigma_{ij} = \partial f_{el}/\partial u_{ij}$ holds, because Maxwell relations are violated; e.g., $\partial\sigma_{12}/\partial u_{22} \neq \partial\sigma_{22}/\partial u_{12} \neq 0$.)

Basic to our approach is the assumption that a finite elastic region exists. This is universally accepted in soil mechanics, eloquently supported by de Gennes [7], and in fact corroborated by the stability of sandpiles. There are of course well argued reservations, such as those derived from force arches, or from the distinction between plastic versus elastic contacts—introduced partly due to the same desire to understand yield [8]. However, they are usually based on an "intergranular" or microscopic point of view, and the connection to the elasticity of macroscopic, continuous media is less than clear [9]. A different argument is the possible lack of a unique displacement field U_i. In our opinion, elasticity is, at its core, a robust theory: In spite of crystal defects, frequently rendering U_i ill defined, elasticity remains valid in solids, accounting for its capability to sustain shear stresses—as long as the defects are stationary. By merely standing, sandpiles demonstrate the same capacity, and there is no reason why a carefully constructed elastic theory should not be able to account for all its static, macroscopic behavior as well. After all, whether a proposed theory does exactly this should be its ultimate test.

In the following, a self-contained elastic theory capable of accounting for mechanical yield and volume dilatancy is developed. First, we choose an equilibrium state of arbitrary (thermodynamic) temperature T and packing density ρ_c, with no external forces, especially gravitation, but with the atmospheric pressure present [10]. This virtual state (in spite of its marginalized stability and individually compressed grains) is taken as our system of reference, with a vanishing displacement field U_i. The associated free energy density is $f_1\rho/m$, where $f_1(T)$ is the free energy, and m the mass, per grain. Turning on the gravitation and applying external forces will further strain the granular material and lead to density change. If the force is small and applied slowly enough, this change is elastic,

$$\delta \equiv 1 - \rho_c/\rho = -u_{nn}. \tag{4}$$

As gravitation and normal stress cram the grains, they lead to finite contact areas between them, and give rise to finite elastic moduli. The free energy density becomes

$$f = (f_1/m)\rho + \tfrac{1}{2}K_b u_{nn}^2 + K_a u_{k\ell}^0 u_{k\ell}^0 + \rho G z, \tag{5}$$

$$K_b = \tilde{K}_b \delta^b, \qquad K_a = \tilde{K}_a \delta^a, \tag{6}$$

with $\tilde{K}_a, \tilde{K}_b > 0$ for $\delta \geq 0$, $\tilde{K}_b, \tilde{K}_a = 0$ for $\delta < 0$. (The particles lose contact with one another for $\delta < 0$.) Linear elasticity corresponds to $a, b = 0$, while Hertz contacts imply $a, b = 1/2$. However, we did not find a watertight general reason requiring $a = b$, as assumed in the quasielastic theory [7]. We believe this is a question of clarity versus accuracy: Experiments are better accounted for if b is taken slightly larger than a (see below). Yet a is the rather more important exponent, which alone already gives rise to mechanical yield and volume dilatancy. So b need not be treated with the same care, and one may set $b = a$ to gain great simplifications in the expressions, mainly because the Poisson ratio

$$\nu \equiv (3K_b - 2K_a)/(6K_b + 2K_a) \tag{7}$$

then lacks critical density dependence. Finally, Eq. (6) should be taken in the spirit of an expansion that holds only close to ρ_c, or for $\delta \ll 1$.

Given Eqs. (5) and (6), and employing the Eulerian notation, $2u_{ij} = \nabla_i U_j + \nabla_j U_i - \nabla_i U_k \nabla_j U_k$, the stress is determined by energy and momentum conservation (cf. [11]) to be $\sigma_{ij} = (\rho\mu - f)\delta_{ij} - \Psi_{ij} + \Psi_{ik}u_{kj} + \Psi_{jk}u_{ki}$, where $\Psi_{ij} \equiv (\partial f/\partial u_{ij})_{\rho,T}$ and $\mu \equiv (\partial f/\partial \rho)_{u,T}$. Keeping only the dominant of the nonlinear terms, we have

$$\begin{aligned}\sigma_{ij} = &- K_b u_{nn}\delta_{ij} - 2K_a u_{ij}^0 \\ &+ \delta^{-1}(\tfrac{1}{2}bK_b u_{nn}^2 + aK_a u_{\ell k}^0 u_{\ell k}^0)\delta_{ij}.\end{aligned} \tag{8}$$

Setting $a, b = 0$, only the first line, or the expression of linear elasticity, remains. The next line comes either from deriving K_a, K_b with respect to $\rho = \rho_c(1 - \delta)$, leading to a contribution in the chemical potential μ, or equivalently, with respect to $u_{nn} = -\delta$, giving rise to additional terms in Ψ_{ij}. Although $a = b \neq 0$ in the quasielastic theory [7], neither does its stress contain the second line. As will be shown in a future work, this is qualitatively alright for silos, because yield is never a problem here.

To obtain a feeling for the implications of Eq. (8), consider the pressure $P \equiv \sigma_{kk}/3$, as a function of the compression δ and the shear $u_s \equiv \sqrt{u^0_{\ell k} u^0_{\ell k}}$,

$$P = (1 + b/2)\tilde{K}_b \delta^{1+b} + a\tilde{K}_a u_s^2/\delta^{1-a}. \tag{9}$$

δ is plotted versus P for given u_s and realistic values of a, b, in the left-hand side of Fig. 1. The solid part of the lines is the stable, physical region, with a positive compressibility; the dashed lines, showing $P \to \infty$ for $\delta \to 0$, are the unstable, unphysical region. The reason for this can be understood from the right-hand side of Fig. 1, a plot of δ versus u_s for given P. Starting at the top, with finite compression and no shear, $\delta = 4 \times 10^{-4}$, $u_s = 0$, we see how the solid lines decrease, or how volume dilates, for increasing shear. (Conventional, linear elasticity yields a straight horizontal line, stopping at the value asserted by the Coulomb yield condition.) Shear values to the right of the parabolalike curves obviously do not have elastic solutions, though as mentioned plastic flow solutions of course must exist.

Thermodynamic stability, however, is lost even before the vanishing of the elastic solutions, at the points given by the arrows, where the solid lines turn into dashed lines. This is because f of Eq. (5) is convex only for certain values of δ and u_s: With $f = \frac{1}{2}\tilde{K}_b \delta^{b+2} + \tilde{K}_a \delta^a u_s^2 + \cdots$, thermodynamic stability requires $(\partial^2 f/\partial \delta^2)(\partial^2 f/\partial u_s^2) \geq (\partial^2 f/\partial \delta\, \partial u_s)^2$, or

$$u_s^2/\delta^2 \leq (2 + b)(1 + b)K_b/[2a(1 + a)K_a]. \tag{10}$$

The dashed lines of both figures represent parameters that do not satisfy this condition. Note that if the medium is not at all compressed, no finite shear is stable.

For further comparison of this theory to the Coulomb yield condition (1), we shall in the following consider three typical experimental setups (see upper inset of Fig. 2). (i) Simple shear test: An infinite layer of sand subject to a normal and a shear force density, N and S (ii) Axisymmetric triaxial test: A cylindrical sample of sand subject to a hydrostatic pressure p and a deviatory normal stress q. (iii) Sand on a slope: An infinite layer of sand on a rough, inclined plane with the angle θ. The implications of Eq. (1) for these experiments are, respectively [1],

$$S/N \leq \tan\varphi_1; \qquad q/(2p + q) \leq \sin\varphi_2; \qquad \theta \leq \varphi_1, \tag{11}$$

with $\varphi_1 = \varphi_2$ denoting the Coulomb angle. Using the above granular elastic theory, setting for simplicity $a = b$, the respective results may also be dressed as the same inequalities, though φ_1, φ_2 are now explicitly given,

$$\tan\varphi_1 = \frac{\sqrt{3(1 - 2\nu)(5a\nu + 2\nu - a + 2)}}{\sqrt{2a}(a\nu - 2\nu + a + 4)}, \tag{12}$$

$$\sin\varphi_2 = \frac{3\sqrt{1 - 2\nu}}{2\sqrt{2a(2 + a)(1 + \nu)} + \sqrt{1 - 2\nu}}. \tag{13}$$

These expressions are easily derived. For the simple shear test, the symmetry of the geometry and the force balance $\nabla_k \sigma_{ik} = 0$ (neglecting gravity) prescribe constant strain, with the displacement given as $U_x, U_y \sim y$, $U_z = 0$. Inserting the nonvanishing components of the strain, u_{xy} and $u_{yy} = -\delta$, into Eq. (8), we obtain

$$\sigma_{yy} = (1 + \tfrac{b}{2})K_b \delta + \tfrac{4}{3}(1 + \tfrac{a}{2})K_a \delta + 2aK_a u_{xy}^2/\delta, \tag{14}$$

$$\sigma_{xy} = -2K_a u_{xy}, \qquad \sigma_{xx} = \sigma_{yy} - 2K_a \delta. \tag{15}$$

The boundary conditions impose $\sigma_{yy} = N$, $\sigma_{xy} = S$. Solving δ and u_{xy} as functions of N and S, we find that no solution exists if N/S exceeds a maximal value. In addition, δ and u_{xy} must satisfy Eq. (10), which for the present case is the more stringent one, leading to Eq. (12). Note that because Eq. (14) is similar to Eq. (9) (they are structurally identical for $a = b$), it also displays dilatancy.

The strain is again constant for the axisymmetric triaxial test. With the displacement vector given as $U_x \sim x$, $U_y \sim y$, $U_z \sim z$, the strain is $u_{xx} = u_{yy} = -(\delta +$

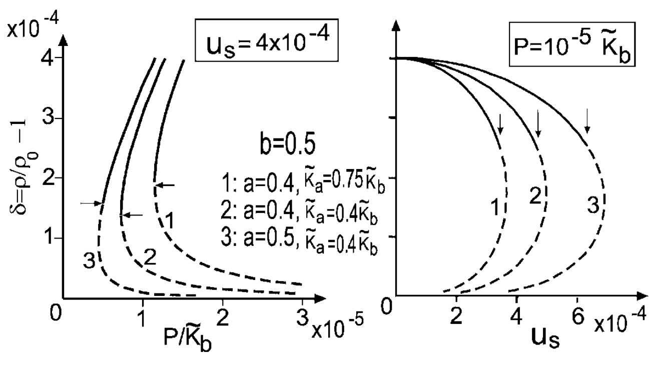

FIG. 1. Compression δ versus pressure P (left) and shear u_s (right). Stability is maintained only where the lines are solid.

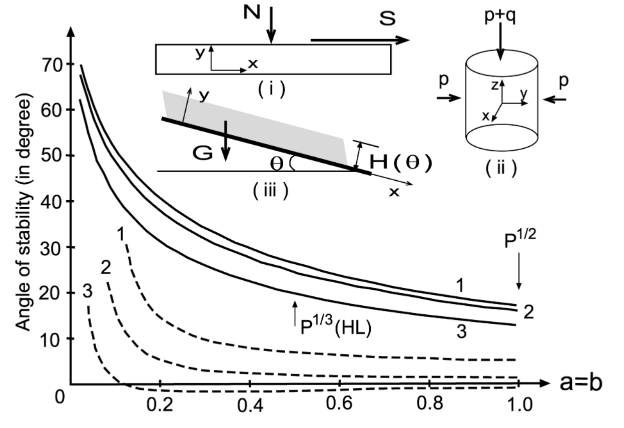

FIG. 2. Plots of φ_1 (solid lines) and $\varphi_2 - \varphi_1$ (dashed lines) as functions of a for $\nu = 0.2, 0.3, 0.4$ (denoted as 1, 2, 3).

$\Delta H/H_0)/2$, $u_{zz} = \Delta H/H_0$, and $u_{ij} = 0$ for $i \neq j$. (H_0 is the height for $p = q = 0$, and ΔH its change.) Inserting these into Eq. (8) yields the stress tensor,

$$\sigma_{xx} = \sigma_{yy} = \tfrac{a}{6}q^2/(K_a\delta) - \tfrac{1}{3}q + K_b(1 + \tfrac{b}{2})\delta,$$
$$\sigma_{zz} = \sigma_{yy} - (\delta + 3\Delta H/H_0)K_a. \tag{16}$$

The boundary conditions are $\sigma_{xx} = \sigma_{yy} = p$, $\sigma_{zz} = p + q$, $\sigma_{ij} = 0$ for $i \neq j$. Again, δ and ΔH do not have solutions if $q/(2p + q)$ exceeds the value given by Eq. (13). [Equation (10) yields exactly the same constraint here.]

Because of high external forces, the actual deformations in both above experiments tend to contain considerable plastic contributions, invalidating a direct comparison with the present theory. This is not the case for the third experiment, sand on a slope, as its deformation is due to the comparatively small gravity. The medium is uniform along the x and z directions (see inset of Fig. 2), so the displacement is of the form $U_x = U_x(y)$, $U_y = U_y(y)$, $U_z = 0$. The stress tensor is then given by Eq. (8) and by integrating $d\sigma_{yy}/dy = -\rho G\cos\theta$, $d\sigma_{xy}/dy = \rho G\sin\theta$ under the constraint that the stress vanishes at the free surface ($y = H$),

$$\sigma_{xx} = (1 + \tfrac{b}{2})K_b\delta + \tfrac{2}{3}(a - 1)K_a\delta + 2aK_a u_{xy}^2/\delta,$$
$$\sigma_{xy} = -2\mu u_{xy} = -GM(y)\sin\theta, \tag{17}$$
$$\sigma_{yy} = \sigma_{xx} + 2K_a\delta = GM(y)\cos\theta.$$

$M(y) \equiv \int_y^H \rho(y')dy'$ denotes mass per unit area between y and the free surface H; the total mass $M(y = 0)$ is taken as a constant that does not vary with θ, so both ρ and H depend on θ. As in the simple shear test, the stability condition, Eq. (10), is the more stringent one confining the value of $\sin\theta$ and leads to Eqs. (12).

Finally, we consider what these results imply for the parameters of the above granular elastic theory, especially the powers a, b and the Poisson ratio ν. In Fig. 2, φ_1 and $\varphi_2 - \varphi_1$ are plotted as functions of a with different values of ν, rendered in solid and dashed lines, respectively. The angle φ_1 varies between 15° and 35°, with $|\varphi_2 - \varphi_1|/\varphi_1 \lesssim 5\%$ in the parameter regime $0.2 < a < 1$ and $0.35 < \nu < 0.4$. Insisting on $\varphi_2 = \varphi_1 = 30°$ for Coulomb media, we find $a = 0.27$ and $\nu = 0.36$. (The fact that $|\varphi_2 - \varphi_1|$ diverges for small a indicates that linear elasticity, $a \to 0$, is incompatible with a unique angle of friction.) Because φ_1 is the maximum angle for inclined planes with a layer of sand at rest, we may identify φ_1 with Bangold's maximum angle of stability. Experimentally, this angle is found to vary from 30° (spherical grains) to 60°–70° (angular and rough grains) [12]. According to Fig. 2, this implies an a between 0 and 0.4.

Because we assumed $b = a$, b also varies between 0 and 0.4 [cf. Eqs. (6) and (9)], we have β varying between 0 and 0.28—smaller than the $\frac{1}{3}$ of the Hertz contact [6]. This is an indication that $a = b$ oversimplifies. Repeating the above calculation allowing $a \neq b$, we find $\varphi_1 = \varphi_2 = 30°$ by taking $a = 0.4$, $b = 0.5$ (i.e., $\beta = \frac{1}{3}$), and $\tilde{K}_a/\tilde{K}_b = 0.36$.

At higher pressures (from 700 to 7000 kg/m^2), $\beta = 0.5$ (or $b = 1$) was measured and referred to as the $P^{1/2}$ dependence [13]. Various microscopic reasons were proposed for this deviation from the Hertz law [14,15]. Within the present framework, this is easily accounted for by the packing dependence of $b(\rho_c)$, a feature that we shall study in future works.

It is important to realize that the Poisson ratio ν as given in Eq. (7) is, in granular materials, not the same as that given by

$$\nu_{\mathrm{tri}} = -\left(\frac{du_{xx}/dq}{du_{zz}/dq}\right) = \frac{1}{2}\frac{b^2 + 3b + (b^2 + 3b + 6)\nu}{b^2 + 3b + 3 + b(b + 3)\nu} \tag{18}$$

(at $q \to 0$), e.g., measured in triaxial experiments. [The second expression is obtained by employing Eq. (16).] Taking $b = 0.5$, $a = 0.4$, $\tilde{K}_a/\tilde{K}_b = 0.36$ as above, and $\delta = 10^{-5}$, we obtain $\nu_{\mathrm{tri}} = 0.25$, well comparable to the measured value of around 0.17 to 0.25 [16]. (Enforcing $a = b$ again leads to discrepancy: Taking $a = b = 0.27$ and ν=0.36 as given above, we have $\nu_{\mathrm{tri}} = 0.4$.)

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