Ballistic Diffusion Induced by a Thermal Broadband Noise

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(Received 23 December 2002; published 26 September 2003)

We present a thermal broadband noise from the difference between two Ornstein-Uhlenbeck noises, which can induce a ballistic diffusion, i.e., long-time mean square displacement of a free particle driven by this noise reads $\langle x^2(t)\rangle \propto t^2$. We apply this noise to a flashing ratchet and the mean velocity of the particle is calculated via Langevin simulation. The results show that a double peak of the mean velocity and flux reversal appears for the ratchet with large and small asymmetries, respectively; the inertia effect induces a large mean velocity and multireversal of flux. These rich and interesting phenomena are explained.

DOI: 10.1103/PhysRevLett.91.138104 PACS numbers: 87.15.Vv, 05.40.Fb, 87.10.+e

In recent years, growing attention has been paid to the processes that take place in disordered media and other systems which show anomalous diffusive behaviors [1]. One of dynamical origin of the anomalous diffusion is due to nonlocality in time and thus the velocity of the particle shows a memory effect, resulting in a non-Markonian Langevin equation (NMLE) [2–4]. Morgado *et al.* [5] modified the spectral density of the thermal bath by removing the low-frequency part of the acoustic modes, leading to a strong superdiffusion, i.e., the mean square displacement of a particle in the force-free case can reach as $\langle x^2(t)\rangle \propto t^{1.98\pm0.01}$. Superdiffusion ($\langle x^2(t)\rangle \propto t^{\mu}$ with $1<\mu<2$) has been observed in a number of systems [6] ranging from early discoveries in intermittent chaotic systems, fluid particles in fully developed turbulence and to millennial climate changes. Unfortunately, only a speculative fractal spectrum, i.e., non-Ohmic friction, was used frequently to discuss anomalous diffusion caused by nonlocality in time, there lacks a practical example of a thermal noise-induced superdiffusion. For instance, it has not been found that the thermal harmonic noise can induce the superdiffusive behaviors. Moreover, the operational mechanisms of molecular motors and the particle separation have been studied extensively and our knowledge about the mode of operation of molecular motors has greatly improved since the first ratchet models [7]. However, how superdiffusion helps directed transport remains as an open question [8].

In this Letter, we present a thermal broadband noise leading to the ballistic diffusion (i.e., $\langle x^2(t)\rangle \propto t^2$ in the case of both force-free and long-time limit) and study the operation of a periodically flashing sawtooth ratchet subjected to such noise. The mean velocity of the particle is calculated via numerical simulation of a set of Markovian Langevin equations (MLE) transformed from the original NMLE. The directed motion of the particle shows some novel behaviors compared with that of the previous results based on white and other colored noises.

The NMLE for the motion of a particle in a potential U reads

$$m\ddot{x}(t) + m\int_0^t \eta(t-t')\dot{x}(t')dt' + U'(x) = \varepsilon(t) + E(x,t), \tag{1}$$

where $\eta(t)$ is the friction memory kernel, $\varepsilon(t)$ is a thermal colored noise that we assume to be zero centered, stationary, and its correlation function obeys the fluctuation-dissipation theorem: $\langle\varepsilon(t)\varepsilon(t')\rangle = k_BT\eta(|t-t'|)$, where k_B is the Boltzmann constant, T is the absolute temperature of the environment, and $E(x,t)$ is a multiplicative noise.

As a practical example, we report a thermal or internal broadband colored noise, which allows a coverage from "red" noise to "green" noise [9,10] associated with a memory kernel as

$$\eta(t-t') = A\left[\frac{1}{\tau_2}\exp\left(-\frac{|t-t'|}{\tau_2}\right) - \frac{1}{\tau_1}\exp\left(-\frac{|t-t'|}{\tau_1}\right)\right], \tag{2}$$

with $A = \eta_0\tau_1^2/(\tau_1^2-\tau_2^2)$, where τ_1 and τ_2 are two correlation times, η_0 is the friction coefficient. According to the Wienerr-Khintchine theorem, the spectral density of noise is the Fourier transformation of the correlation function of noise,

$$S(\omega) = k_BT\frac{2\eta_0\tau_1^2\omega^2}{(1+\tau_1^2\omega^2)(1+\tau_2^2\omega^2)}. \tag{3}$$

Indeed, this noise can be realized from the difference between two Ornstein-Uhlenbeck noises (OUN) with different time constants driven by the same white noise. Alternatively, one can use a red noise to excite a series circuit or resistance and capacitance, and then the noise needed is finally obtained from the voltage of the resistance. Thus the present broadband noise is also a second-order colored noise. Note that the low-frequency part of this noise has been removed [$S(0) = 0$] and the spectrum (3) shows a band-passing behavior. This implies that the effective friction is vanishing in long-time limit, i.e., $\lim_{t\to\infty} \int_0^t \eta(t - t')dt' = 0$. This is why one can expect the ballistic behavior to emerge.

We first consider a case that both potential trapezoid and external driving are absent, thus the solution of NMLE (1) can be obtained by means of the Laplace transform technique,

$$x(t) = x_0 + v_0 H(t) + \int_0^t H(t - t')\varepsilon(t')dt', \quad (4)$$

where x_0 and v_0 are the initial position and velocity of the particle. The response function $H(t)$ is the inverse form of the Laplace transform $\hat{H}(s) = [s^2 + s\hat{\eta}(s)]^{-1}$, where $\hat{\eta}(s) = \int_0^\infty \eta(t)\exp(-st)dt$ is the Laplace transformation of the friction memory kernel (2), we have

$$\hat{H}(s) = \frac{(1 + s\tau_1)(1 + s\tau_2)}{s^2\{\eta_0\tau_1^2/[(\tau_1 + \tau_2) + (1 + s\tau_1)(1 + s\tau_2)]\}}. \quad (5)$$

Applying the residue theorem, we obtain the response function $H(t)$ and then the mean square displacement is yielded as

$$\langle x^2(t)\rangle_{\mathrm{free}} = \frac{k_B T}{m}\left\{D_N 2t + D_S t^2 - \frac{a^2(\tau_1 + \tau_2)^2}{(1 + a)^4} + 2\int_0^t \Xi(t')dt' - \Xi^2(t) - 2\left[\frac{t}{1 + a} + \frac{a(\tau_1 + \tau_2)}{(1 + a)^2}\right]\Xi(t)\right\}, \quad (6)$$

where $D_N = (\tau_1 + \tau_2)a^2/(1 + a)^3$, $D_S = a/(1 + a)^2$, $a = \eta_0\tau_1^2/(\tau_1 + \tau_2)$, and the time decay term has the form

$$\Xi(t) = \frac{1}{\tau_1\tau_2(\nu_1 - \nu_2)}\left[\frac{(1 + \nu_1\tau_1)(1 + \nu_1\tau_2)}{\nu_1^2}\exp(\nu_1 t) - \frac{(1 + \nu_2\tau_1)(1 + \nu_2\tau_2)}{\nu_2^2}\exp(\nu_2 t)\right], \quad (7)$$

where ν_1 and ν_2 are the roots of the equation $s^2 + (\tau_1 + \tau_2)(\tau_1\tau_2)^{-1}s + (1 + a)(\tau_1\tau_2)^{-1} = 0$; both real parts of the two roots are negative. When $\tau_1 \to \infty$, $D_S = 0$, and $D_N = \eta_0^{-1}$, the present process becomes an OUN process with corresponding memory kernel. Moreover, if $\tau_2 \to 0$ this will reduce to the regular Brownian motion.

It is worth pointing out that the noise introduced here differs from the harmonic noise [11]; the latter is a narrow-band noise and also called a quasimonochromatic noise, its high-frequency part of the spectrum is declined and low-frequency part does not vanish. Moreover, the fractal noise proposed by Jung [12], in particular, is an overdamped harmonic noise. For the thermal harmonic noise, the friction kernel is $\eta(t) = (\Omega^2\eta_0/\gamma) \times \exp(-\gamma t/2)[\cos\omega_1 t + (\gamma/2\omega_1)\sin\omega_1 t]$ [11], where $\omega_1^2 = (\gamma^2/4 - \Omega^2)$, Ω and γ are the frequency and damping of the oscillator, respectively. Thus the Laplace transformation of the response function in the force-free case is given by $\hat{H}(s) = s^{-1}\{s + (\Omega^2\eta_0/\gamma)(s + \gamma)/[\omega_1^2 + (s + \gamma/2)^2]\}^{-1}$, which will lead to $\lim_{t\to\infty}\langle x^2(t)\rangle \propto t$. Namely, both the harmonic noise and the fractal noise cannot give rise to superdiffusion, because the Laplace transformations of their response functions have no double zero roots as compared to the case of the broadband noise proposed in this Letter.

We now transform the NMLE (1) by describing a particle moving in a flashing ratchet [13,14] [i.e., $E(x, t) = [1 - z(t)]U'(x)$ in Eq. (1)] into a set of the MLE with four variables as

$$\begin{aligned} \dot{x} &= v, \qquad m\dot{v} = -z(t)U'(x) + y_1(t) + y_2(t),\\ \dot{y}_1 &= -\frac{1}{\tau_1}y_1(t) + \frac{A}{\tau_1}v(t) - \frac{1}{\tau_1}\xi(t),\\ \dot{y}_2 &= -\frac{1}{\tau_2}y_2(t) - \frac{A}{\tau_2}v(t) + \frac{1}{\tau_2}\xi(t), \end{aligned} \quad (8)$$

where $\xi(t)$ is a white noise with $\langle\xi(t)\rangle = 0$, $\langle\xi(t)\xi(t')\rangle = 2D\delta(t - t')$, and $D = \eta_0 k_B T[\tau_1/(\tau_1 - \tau_2)]^2$; $z(t)$ takes two values 0 and 1, as it is a periodical dichotomous process, the particle experiences the potential off during the waiting times t_{off} and the potential on during t_{on}, we have

$$z(t) = \begin{cases} 0, & nt_p < t < nt_p + t_{\mathrm{off}};\\ 1, & nt_p + t_{\mathrm{off}} < t < (n + 1)t_p, \end{cases} \quad (9)$$

where $t_p = t_{\mathrm{off}} + t_{\mathrm{on}}$.

In Fig. 1, we plot the numerical results of the mean square displacement $\langle x^2(t)\rangle$ in the force-free case by means of Eq. (8) with $U' = 0$. The error of the present algorithm related to theoretical data [Eq. (6)] is less than 1%. It is seen that $\langle x^2(t)\rangle$ is proportional to t^2 in the long-time limit, namely, the ballistic diffusion is observed when the noise is in the region of green noise for small-τ_2 and finite-τ_1.

Now we apply the proposed thermal broadband noise to a periodically flashing ratchet model. Although this is a very simple model which is not enough for describing the operation of the real molecular motor, it is very tutorial and intuitive. The flashing ratchet with a normal velocity-memory has been studied in Ref. [15]. The ratchet potential is chosen to be a simple sawtooth one,

$$U(x) = \begin{cases} \frac{U_0}{(1-\alpha)\lambda}x, & k\lambda \le x \le (k + 1 - \alpha)\lambda;\\ \frac{U_0}{\alpha\lambda}(\lambda - x), & (k + 1 - \alpha)\lambda \le x \le (k + 1)\lambda, \end{cases}$$

where U_0 is the barrier height, while α and λ denote the

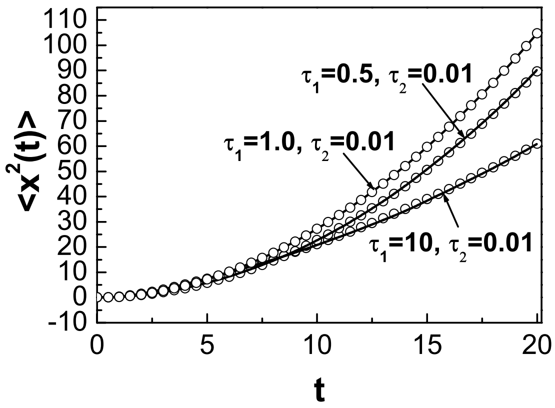

FIG. 1. The mean square displacement of a free particle vs time for various τ_1 and τ_2 at the temperature $k_BT = 1.0$. The solid lines and open circles are theoretical and numerical results, respectively.

asymmetric parameter and the periodic length of the ratchet potential. For ratchet model, we rescale the position and time [16], and thus the new dimensionless dynamics can be obtained; here the rescaled mass and noise parameters becomes dimensionless variables. The probability current J is related to the mean velocity $\langle v \rangle$ in the stationary state: $J = \langle v \rangle / \lambda$. The steady mean velocity of the particle is evaluated numerically by $\langle v \rangle = [\langle x(t) \rangle - x(0)]/t$.

In Fig. 2, we plot the mean velocity as a function of the half cycle period $t_p/2$ for two kinds of asymmetrical parameters $\alpha = 0.6$ and $\alpha = 0.9$, here and below we have chosen $t_{\text{off}} = t_{\text{on}} = t_p/2$. The behavior of the mean velocity is under the influence of a cooperation and competition between two nonequilibrium processes: diffusion and mobility. A particle is easily diffused at least a short distance but not a long distance while the potential is off; and the particle moves along the sawtooth sides when the potential is on according to the generalized Einstein

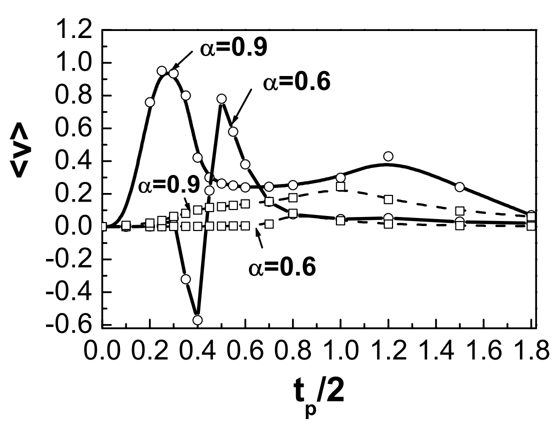

FIG. 2. Mean velocity $\langle v \rangle$ as a function of the half cycle period $t_p/2$ for various α. The solid and dashed lines are the results of ballistic and normal diffusions, respectively. The parameters used are $\tau_1 = 0.5$, $\tau_2 = 0.05$, and $k_BT = 0.01$.

relation $\langle x(t) \rangle_{F_0} = F_0 \langle x^2(t) \rangle_{\text{free}} / (2k_BT)$, where F_0 are two slopes of the ratchet potential. Now if the asymmetry of the ratchet is large, freely diffusive particle can reach the position of the nearest barrier during a short period, however, there is a small possibility for the particle arriving at the position of the further barrier, thus the difference between the forward P_f and backward P_b probabilities can become maximum for a short period of the cycle time, and accordingly the first peak of the mean velocity is observed. With the increase of the cycle time period, the net probability for the drift decreases, but when the particle has enough time to descend to the bottom of a well from the top along the longer side of the potential while the potential is reconverted, thus the particle can move directly a long distance after a cycle period, this leads to the appearance of the second peak. If the asymmetry of the ratchet decreases, the particles will descend first to the bottom of a well along the longer side after they diffuse freely to the farther barrier while the potential is off, so it is possible that the velocity due to the mobility exceeds over that due to the diffusion, leading to the mean velocity become negative. Moreover, the maximum of the mean velocity driven by a thermal white noise appears in a long cycle period, because in the normal diffusion case the variance of the distribution changes very little, thus the particle needs a long time to reach the nearest barrier.

The finite inertia causes the complex behaviors of the directed motion as shown in Figs. 3(a) and 3(b). A global maximum with respect to both m and t_p appears for finite mass [Fig. 3(a)]. Thereby, a novel unexpected feature is found that for small mass the first positive maximum of the mean velocity with respect to $t_p/2$ appears in the region of the small values of $t_p/2$, while the second positive peak shifts toward the large value of $t_p/2$ when the inertial mass increases. A second novel feature is that a negative velocity appears for moderate t_p. First, for moderate values of m we have found a double reversal of the velocity direction for increasing values of the cycle period [Fig. 3(b)]. For larger m the flux exhibits a single reversal and it remained positive for very large $t_p/2$. Both diffusion and mobility have a stronger or weaker effect in small or larger mass, respectively, thus the mobility along the two smooth sides of the ratchet potential exceeds the free diffusion at a moderate m. However, all the above behaviors do not occur in the case of normal flashing ratchet, although a double reversal of the velocity has been observed in the inertia correlation ratchet [16].

In summary, we have proposed and realized a thermal band-passing noise which can lead to a strong superdiffusion, i.e., ballistic diffusion. Indeed, there does not exist the superdiffusive behavior for any other second-order colored noise sources, such as the harmonic noise and the fractal noise. For a flashing ratchet subjected to this noise, the mean velocity of the particle shows rich behaviors in comparison with the normal case. There exists a

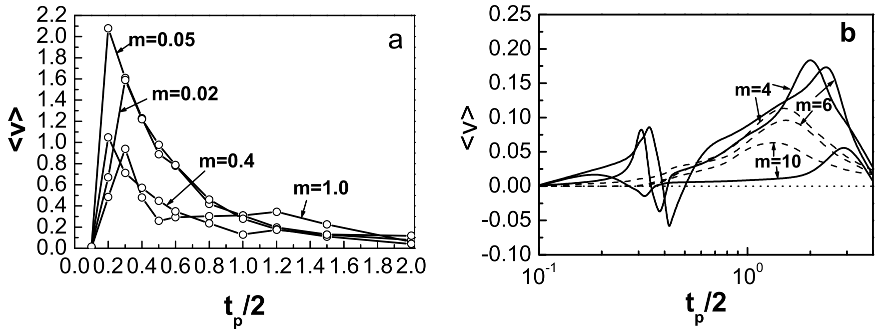

FIG. 3. Mean velocity $\langle v \rangle$ vs the half cycle period $t_p/2$ for various rescaled mass m. The parameters used are $\tau_1 = 0.5$, $\tau_2 = 0.05$, $\alpha = 0.9$, and $k_B T = 0.001$. (a) Small-m case for ballistic diffusion; (b) Large-m case, solid and dashed lines are the results of ballistic and normal diffusions.

double peak and reversal for the mean velocity, this is due to the combined effects of the ballistic diffusion in the potential off and the mobility in the potential on to play the important roles at different time scales, respectively. The maximum of the mean velocity in this superdiffusion case is much larger than that in the normal diffusion. We also demonstrate that the finite inertia enhances the mean velocity for this transport and there exists a flux reversal in the underdamped case. Thus an anomalous Brownian motor in the presence of ballistic diffusion proposed in this work can produce considerably large output work.

It is expected that the ballistic diffusion might become a useful operational mechanism to achieve a high efficiency for the molecular motors and other systems in addition to what have been discussed in Ref. [17]. We believe that the transport produced by the broadband noise can be applied to various problems in the future. Moreover, we conjecture that the superdiffusion with an exponent $1 < \mu < 2$ can be obtained from the difference between two fractional Ornstein-Uhlenbeck noises, or, a white noise minus a fractional OUN, the latter is the solution of a linear Riemann-Liouville fractional differential equation driven by a Gaussian white noise [18]; as well as a double-band-passing noise from the difference between two harmonic noises driven by the same white noise can also be obtained along the line of the present work. This requires one to study further.

This work was supported by the National Natural Science Foundation of China under Grants No. 10075007 and No. 10235020.

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