Kelvin-Wave Cascade on a Vortex in Superfluid ^{4}He at a Very Low Temperature

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A study by computer simulation is reported of the behavior of a quantized vortex line at a very low temperature when there is continuous excitation of low-frequency Kelvin waves. There is no dissipation except by phonon radiation at a very high frequency. It is shown that nonlinear coupling leads to a net flow of energy to higher wave numbers and to the development of a simple spectrum of Kelvin waves that is insensitive to the strength and frequency of the exciting drive. The results are likely to be relevant to the decay of turbulence in superfluid ^{4}He at very low temperatures.

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It is well known that quantized vortices can be formed in superfluid ^{4}He. Such vortices can support a transverse and circularly polarized wave motion (a Kelvin wave), with the approximate dispersion relation for a rectilinear vortex [1]

$$\omega = \frac{\kappa k^2}{4\pi}\left[\ln\left(\frac{1}{ka}\right)+c\right], \tag{1}$$

where κ is the quantum of circulation (h/m_4), a is the vortex core parameter, and $c \sim 1$. The existence of these waves in an inviscid fluid was first discussed as a theoretical possibility in the 19th century [2], but an experimental study in a fluid without viscosity had to await the discovery of quantized vortices in superfluid ^{4}He. Kelvin waves in uniformly rotating superfluid ^{4}He were first observed experimentally by Hall [3], and a number of interesting experimental and theoretical studies have been published subsequently; see, for example, Glaberson *et al.* [4] on an instability in the presence of an axial flow of the normal fluid; and the study of nonlinear effects, leading to soliton behavior [5] and to an associated sideband instability [6]. Nonlinear effects remain interesting, and an aspect of them that is important in our understanding of Kelvin waves at very low temperatures, and which may be of rather general interest in nonlinear dynamics, is discussed in this Letter.

At temperatures where there is a significant fraction of normal fluid, Kelvin waves in superfluid ^{4}He are damped by mutual friction, which is the frictional force exerted on a vortex when it moves relative to the normal fluid. This Letter is concerned with the expected behavior of Kelvin waves at very low temperatures, when damping due to mutual friction can be neglected. Under these conditions Kelvin waves can be damped only by radiation of phonons, but the damping is expected to be extremely small [7] unless the frequency is very large, typically of the order of 4 GHz ($k \sim 2$ nm^{-1}). Kelvin waves of lower frequency are essentially undamped. In these circumstances Kelvin waves of a particular low frequency can lose energy only by nonlinear coupling to waves of a different frequency. We are led therefore to consider the following situation. Suppose that we have a rectilinear vortex of finite length, and that we continuously drive one low-frequency Kelvin mode on this vortex. The mode will grow in amplitude, until nonlinear effects give rise to a transfer of energy to other modes, particularly at higher frequencies. This process will presumably continue until modes are excited that have a frequency sufficiently high for effective phonon radiation. The aim of this Letter is to predict the details of this process, by means largely of computer simulations. We shall consider, for example, whether there is a steady state, in which energy is injected at a low frequency and dissipated by phonon emission at a high frequency, and whether there is some well-defined and simple spectrum of the Kelvin waves existing in this regime. We shall regard this regime as a cascade, although we are not sure whether it is a strict cascade in the sense that energy is transferred *in steps* to higher wave numbers. We shall find that a well-defined spectrum does seem to exist, that it is simple in form, and that it is remarkably insensitive to the amplitude and frequency of the drive.

This behavior is interesting in its own right. However, we ourselves were led to investigate it by an interest in the decay of turbulence in superfluid ^{4}He at very low temperatures [8], where mutual friction has a negligible damping effect on vortex motion. As in classical turbulence, energy in superfluid turbulence must probably flow from larger to smaller length scales, and it has been suggested that on the smallest scales the relevant motion is a Kelvin wave on a vortex with wave number greater than the inverse vortex spacing. It is then of interest to understand how energy can flow in a system of Kelvin waves toward higher wave numbers, where it can ultimately be dissipated at the highest wave number by phonon radiation. In the context of superfluid turbulence the details are likely to be quite complicated (Kelvin waves are likely to be generated by vortex reconnections), and it is not yet generally accepted that a

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Kelvin-wave cascade is strictly necessary. Nevertheless it seems likely that such a cascade has a significant role. These matters have been discussed in a recent review [8], and we plan further discussion in forthcoming papers.

We believe that the behavior we find may exist also in other types of systems and may therefore have other applications. It should be explained that similar problems to that discussed here have been addressed by other authors [9–12], but they relate to situations that are significantly different, for example, where there is no steady state or where the length of vortex line is constrained not to increase. Thus Araki and Tsubota [11] carried out numerical simulations on an initial configuration in which a vortex ring approaches a rectilinear vortex, Kelvin waves being generated by reconnections when collision takes place. There was no steady input of energy and no obvious dissipation. The numerical simulations of Kivotides *et al.* [10] related to an initial configuration of four vortex rings, Kelvin waves again being generated by reconnections when the rings collide. Again there was no steady input of energy and no obvious dissipation. In both these simulations the authors found some evidence for the development of a spectrum similar to Eq. (6), although the situation is obviously different from that considered in this Letter. The computational work of Nemirovskii *et al.* [12] did deal with a steady state in which waves on a vortex ring were generated by low-frequency noise, although the total length of line was kept artificially constant; the results suggested the existence of a spectrum similar to Eq. (3), although it did not exhibit the insensitivity to the driving conditions that we ourselves find. Generally, then, it seems that this other work has not led to such simple and clearcut results as we report in this Letter, and, as we shall discuss in later papers, it may be less relevant to the decay of superfluid turbulence, especially when, as is often likely to be the case [8], energy flows from a large reservoir associated with large-scale quasiclassical turbulent motion into motion on a scale less than the vortex line spacing.

We consider a model system in which the helium is contained in the space between two parallel sheets, separated by distance $\ell_B = 1$ cm, with a single, initially rectilinear, vortex stretched between opposite points on the two sheets. Kelvin waves can be excited on this vortex, and periodic boundary conditions are applied at each end. Thus the allowed wave numbers of the Kelvin waves are given by

$$k = \frac{2\pi n}{\ell_B}, \tag{2}$$

where n is an integer (> 0). We imagine that one of these modes, with n equal to a small integer n_0, is continuously driven, so that its amplitude tends continuously to increase. As the amplitude increases nonlinear coupling to other modes sets in, and we can expect energy to flow from the mode n_0 to other modes, with both larger and smaller wave numbers. Now we introduce a suitably strong damping for all modes with n exceeding a large critical value n_c. This is intended to mimic the effect of phonon emission, although, because of inevitable computational limitations, it is occurring at a much smaller frequency. Then we ask whether there is a steady state, described by an energy spectrum E_k, in which the energy input to the mode n_0 is balanced by dissipation in the modes with $n > n_c$. We find that such a steady state does seem to exist, and we determine the character of the corresponding energy spectrum. We observe no reconnections.

An important feature of the model lies in the fact that Kelvin modes with wave numbers less than $2\pi/\ell_B$ cannot be excited, so that energy cannot flow to smaller and smaller wave numbers. The relevance of this feature to the decay of superfluid turbulence will be discussed in a later paper.

The simulations are based on the vortex filament model, and they are similar to those described by Schwarz [13] and used in more recent work by one of the authors [14]. The undisplaced vortex lies along the z axis. Its calculated motion is based on the full Biot-Savart law and therefore takes account of both local and nonlocal contributions. The force that drives one mode is of the form $V\rho\kappa \sin(k_0 z - \omega_0 t)$, where $k_0 = 2\pi n_0/\ell_B$, ρ is the density of the helium, and ω_0 is related to k_0 by the dispersion relation (1). Damping at the highest wave number allowed by the resolution of the simulations (1/60 cm) is applied by a periodic smoothing process, the details of which will be described in a later publication; this allows an effective dissipation at the highest wave number that can adapt to the flux of energy through k space arising from the drive.

Figure 1 shows how the total length of line evolves in time after application of the driving force. We see that it

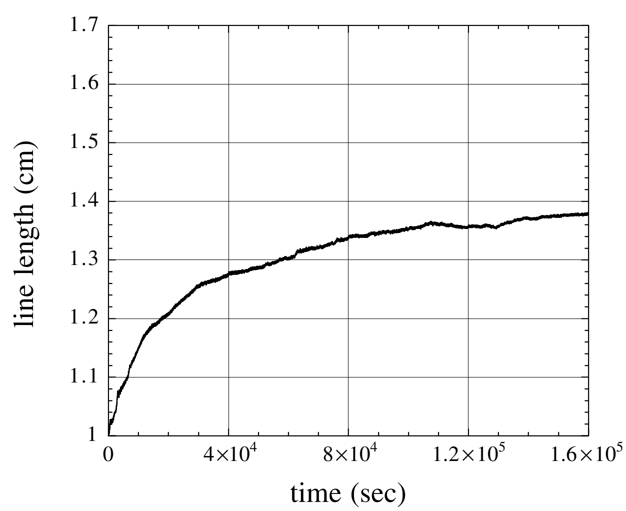

FIG. 1. The development in time of the total length of the vortex line.

reaches a steady average value, suggesting the existence of a steady state.

We express our more detailed results in terms of the root mean square amplitudes $\bar{\zeta}_k(t) = \langle \zeta_k^* \zeta_k \rangle^{1/2}$ of the Fourier components of the displacement of the vortex. Figure 2 shows how these amplitudes develop in time after the application of a drive with $V = 2.5 \times 10^{-5}$ cm s^{-1} and $k_0 = 10\pi$ cm^{-1}. We see that initially only the mode that resonates with the drive is excited. However, as time passes, nonlinear interactions lead to the excitation of all other modes. Eventually the spectrum reaches a steady state, shown by the solid line, and there is then no further change. In this steady state energy is injected at a certain rate at the wave number k_0, and it is dissipated at the same rate at the highest wave number. For large values of k, where the modes form practically a continuum, the steady state is observed to have, to a good approximation, a spectrum of the simple form

$$\bar{\zeta}_k^2 = A \ell_B^{-1} k^{-3}, \tag{3}$$

where the dimensionless parameter A is of order unity.

Figures 3 and 4 show the effects, respectively, of increasing the drive amplitude V by a factor of 10 and of changing the drive wave number k_0. We see that there is no effect on the steady state, within the error of the simulations, at least at the higher wave numbers. The steady state takes longer to be established at the lower drive amplitude, which suggests that even with a small drive amplitude the same steady state would be established after a sufficiently large time, but limitations on the time available for a simulation have not allowed us to check this suggestion.

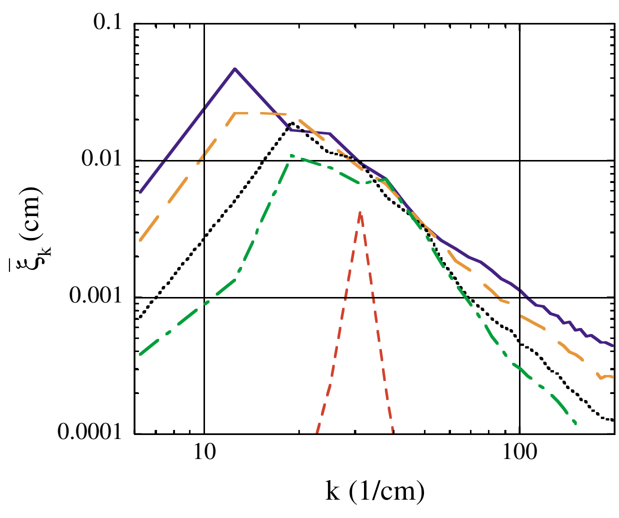

FIG. 2 (color online). Time development of $\bar{\zeta}_k(t)$. The short-dashed line, the dash-dotted line, the dotted line, the long-dashed line, and the solid line refer, respectively, to averages over 0–800, 10 000–10 800, 20 000–20 800, 40 000–40 800, and 140 000–140 800 s. The solid line relates to the steady state.

Important questions relate to the relationship between the drive amplitude and the power input to the system of Kelvin waves. For small times after the drive is first established nonlinearities in the system are relatively unimportant, and the power input can be calculated from the product of the drive amplitude and the amplitude of the velocity response at wave number k_0, taking proper account of phase differences. For later times, however, the system exhibits very strong nonlinear behavior, and this simple technique is no longer applicable. Energy seems to be injected through processes involving two wave vectors, the difference between which is equal to k_0. Therefore we have not yet been able to calculate how the power input varies with the drive amplitude, although we hope to do so later. Careful analysis of the operation of damping in our model for large wave numbers might allow us to obtain the rate of dissipation of energy, but we have not yet completed this analysis. For the present we can only make the reasonable assumption that an increase in the drive amplitude does increase the power input. We can conclude with some confidence, therefore, that the spectrum is insensitive to the drive amplitude, the drive frequency, and the power input from the drive.

The mean energy per unit length of vortex in a mode k is related to $\bar{\zeta}_k$ by the equation

$$E_k = \epsilon_K k^2 \bar{\zeta}_k^2, \tag{4}$$

where ϵ_K is an effective energy per unit length of vortex, given by

$$\epsilon_K = \frac{\rho \kappa^2}{4\pi} \left[\ln\left(\frac{1}{ka}\right) + c_1 \right]. \tag{5}$$

It follows from Eqs. (3) and (4) that

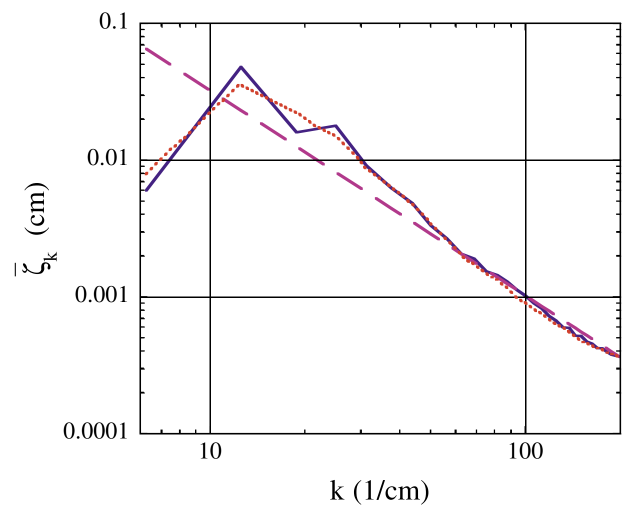

FIG. 3 (color online). Steady-state values of $\bar{\zeta}_k(t)$ for two different drive amplitudes. The solid line and the dotted line are for, respectively, $V = 2.5 \times 10^{-5}$ cm s^{-1} and $V = 2.5 \times 10^{-4}$ cm s^{-1}. The long-dashed line has the form of Eq. (3).

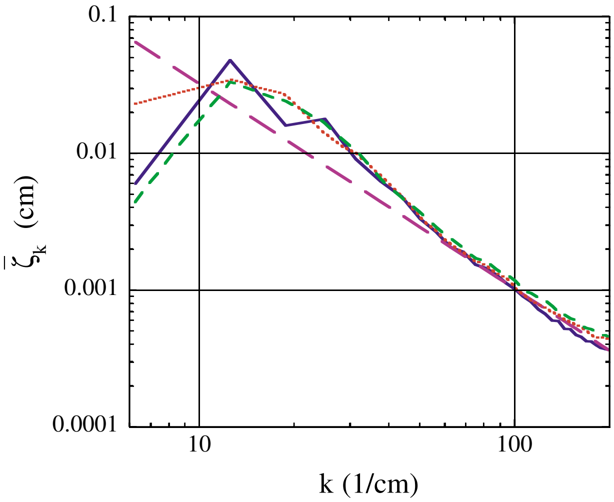

FIG. 4 (color online). Steady-state values of $\bar{\zeta}_k(t)$ for drives at three different wave numbers. The dotted, short-dashed, and solid lines refer, respectively, to $k_0 = 2\pi\ \mathrm{cm}^{-1}$, $k_0 = 4\pi\ \mathrm{cm}^{-1}$, and $k_0 = 10\pi\ \mathrm{cm}^{-1}$. Again the long-dashed line has the form of Eq. (3).

$$E_k = A\epsilon_K (k\ell_B)^{-1}. \tag{6}$$

We conclude that in our model system a steady state cascade does develop, that this state is characterized by the energy spectrum (6), and that, remarkably, this spectrum is insensitive to the frequency and amplitude of the drive and to the power input at the drive frequency.

It is interesting to ask what physics underlies this result. We suggest that, in the steady state and for waves of wave number of order k, there is a saturation in the local amplitude of the Kelvin waves at a value of roughly k^{-1}; this arises from a sudden onset of strong nonlinear effects when the amplitude is of the order of the wavelength. We make the reasonable assumption that the total mean square amplitude of the displacement of the vortex is independent of the length (ℓ_B) of the vortex. The spectrum (3) must then be proportional to ℓ_B^{-1}. Our assumption about the sudden onset of nonlinear effects means that the only other parameter on which this spectrum can depend is k. The form (3) then follows from a dimensional argument. We propose now to investigate whether this type of behavior can be found in other forms of wave propagation, at least if they have the same types of dispersive and nonlinear characteristics (which are known to lead to soliton behavior [5]).

As far as we are aware, there are as yet no experimental results that relate directly to the behavior of Kelvin waves in superfluid ^{4}He at very low temperatures. Such experiments, which might involve the excitation of Kelvin waves on the regular array of vortices existing in the uniformly rotating liquid, as in the early experiments of Hall [3], would be of great interest. Measurements might be made of the rate at which energy is transmitted to the array from a suitable transducer, and of the rate at which energy eventually appears as heat after the Kelvin-wave cascade has been established.

In summary we have reported the results of computer simulations of the behavior of Kelvin waves on a rectilinear quantized vortex of finite length in superfluid ^{4}He at a temperature so low that the waves suffer no attenuation from mutual friction with the normal fluid, the only attenuation arising from phonon radiation at a very high frequency. The waves are excited by continuously driving the system at a small wave number. The amplitude of the driven mode increases until nonlinear coupling leads to a transfer of energy to all other modes. A steady state is established, described by a simple energy spectrum, the form of which is remarkably insensitive to the strength and other details of the drive.

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[1] R. J. Donnelly, *Quantized Vortices in Helium II* (Cambridge University Press, Cambridge, 1991).
[2] W. Thomson, Philos. Mag. **10**, 155 (1880).
[3] H. E. Hall, Proc. R. Soc. London, Ser. A **245**, 546 (1958).
[4] W. I. Glaberson, W. W. Johnson, and R. M. Ostermeier, Phys. Rev. Lett. **33**, 1197 (1974).
[5] H. Hasimoto, J. Fluid Mech. **51**, 477 (1972).
[6] D. C. Samuels and R. J. Donnelly, Phys. Rev. Lett. **64**, 1385 (1990).
[7] W. F. Vinen, Phys. Rev. B **64**, 134520 (2001).
[8] W. F. Vinen and J. J. Niemela, J. Low Temp. Phys. **128**, 167 (2002).
[9] B. V. Svistunov, Phys. Rev. B **52**, 3647 (1995).
[10] D. Kivotides *et al.*, Phys. Rev. Lett. **86**, 3080 (2001).
[11] T. Araki and M. Tsubota, J. Low Temp. Phys. **121**, 405 (2000).
[12] S. K. Nemirovskii, J. Pakleza, and W. Poppe, Russ. J. Eng. Thermophys. **3**, 369 (1993).
[13] K. W. Schwarz, Phys. Rev. B **31**, 5782 (1985); **38**, 2398 (1988).
[14] M. Tsubota, T. Araki, and S. K. Nemirovskii, Phys. Rev. B **62**, 11751 (2000).