Why Are Chaotic Attractors Rare in Multistable Systems?

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We show that chaotic attractors are rarely found in multistable dissipative systems close to the conservative limit. As we approach this limit, the parameter intervals for the existence of chaotic attractors as well as the volume of their basins of attraction in a bounded region of the state space shrink very rapidly. An important role in the disappearance of these attractors is played by particular points in parameter space, namely, the double crises accompanied by a basin boundary metamorphosis. Scaling relations between successive double crises are presented. Furthermore, along this path of double crises, we obtain scaling laws for the disappearance of chaotic attractors and their basins of attraction.

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Many processes in nature do not possess only one long-term asymptotic state or attractor but are rather characterized by a large number of coexisting attractors for a given set of parameters. This phenomenon, called multistability, is found commonly in many fields of science such as neuroscience [1], chemistry [2], optics [3], and condensed matter physics [4].

We consider a general class of smooth nonlinear dynamical dissipative systems which are obtained by taking the conservative limit and adding a small amount of dissipation. Physicists have been commonly studying systems in which the dynamics is either strongly dissipative, such as the Hénon map, or conservative, such as the Hamiltonian systems. But in practice, the systems are more often neither strongly dissipative nor conservative, but they are somewhere in between, i.e., weakly dissipative, as addressed in this Letter. Usually such systems possess many coexisting attractors depending on two control parameters, where one of them can be regarded as a forcing (nonlinearity) and the other one as a damping.

For a given damping, the forcing controls the bifurcation sequence of each individual attractor. In general, these attractors appear through a saddle-node bifurcation; they then undergo a period-doubling cascade that ends up in chaos. The chaotic attractor is destroyed in a boundary crisis [5] in which the attractor collides with an unstable periodic orbit on the boundary of its basin of attraction.

If we fix the forcing but vary the damping, we can change two properties of the system: first, the number of coexisting attractors and, second, the length of the parameter intervals for which the attractors are chaotic. For strong damping, there exists typically only one bounded attractor in a wide range of the parameter space. Moreover, the parameter interval in which this attractor is chaotic is rather wide and interspersed with periodic windows. But the behavior close to the conservative limit, with small damping, is more involved. Since the number of coexisting attractors scales as one per damping [6,7], this number can be made arbitrarily large by choosing a small damping. Among this large number of coexisting attractors, most are periodic and none or very few are chaotic. Additionally, if they happen to be chaotic, their existence is limited to extremely tiny parameter intervals. An important question arising from our studies of multistable systems is the following: how prevalent are chaotic attractors in parameter space?

The aim of this Letter is to show that chaotic attractors are rare in multistable systems. As a main result, we show that *chaotic attractors disappear for two reasons as we approach the conservative limit.* First, the existence intervals in parameter space shrink geometrically with the Feigenbaum constant for conservative systems. Second, their *basins of attraction* in a bounded region of the state space *shrink exponentially* as the damping approaches zero. For a special path in parameter space, which corresponds to double crises accompanied by a basin boundary metamorphosis [8], we obtain scaling laws for the disappearance of chaotic attractors and their basins of attraction. The explanation of the suggested scaling along this particular path in parameter space involves many different phenomena which we describe next using a generic example for an illustration.

To demonstrate the disappearance of chaotic attractors in multistable systems, we use the Hénon map as a paradigm:

$$x_{n+1} = A - x_n^2 - (1 - \nu)y_n, \qquad y_{n+1} = x_n. \quad (1)$$

This system possesses two parameters. The nonlinearity parameter A (forcing) represents the bifurcation or control parameter. The damping parameter ν varies between 0 (conservative limit) and 1 (strong damping).

While for $\nu = 1$ there exists one chaotic attractor in a wide range of the parameter A, there are several coexisting periodic and chaotic attractors for values of ν close to the no damping limit $\nu = 0$. Since all these attractors evolve in the same generic manner, it is sufficient to study

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the bifurcation sequence of *one* of those attractors. We focus on the attractor with the biggest interval of chaotic behavior and the biggest basin of attraction.

To address the question of parameter intervals for the existence of chaotic attractors in weakly dissipative systems, we consider the bifurcation structure in the two parameter space spanned by forcing and damping. We measure the extension of the chaotic region with respect to the forcing parameter A as the damping ν varies. For a given ν value, we define the chaotic region as being bounded by the accumulation point of the period-doubling cascade A_F (Feigenbaum point) where the chaotic attractor appears and the value $A_{\rm cr}$ at the boundary crisis, where the chaotic attractor disappears. Of course we understand that apparently there is a dense set of periodic windows embedded in the chaotic region. As shown in Fig. 1, the length of the chaotic region $\Delta A_{\rm chaos} \equiv A_{\rm cr} - A_F$ asymptotes to zero as we decrease the damping towards the conservative limit. There are points along the curve $\Delta A_{\rm chaos}$ in which the derivative of the curve is discontinuous. These vertices $v_0, v_1, \ldots, v_k, \ldots$ denote special points in parameter space corresponding to double crises [8], where $k \to \infty$ as $\nu \to 0$. Such a double crisis is characterized by a simultaneous occurrence of a boundary crisis, where the chaotic attractor touches its basin boundary, and an interior crisis, where the attractor changes suddenly its size. For the same parameter value a metamorphosis of the basin boundary takes place, where a sudden enlargement of the fractal structure of the basin boundary is observed.

Let us discuss the question of the existence of chaotic attractors in a given parameter interval from another point of view. The probability of finding a chaotic attractor in a multistable system does not depend only on its abundance in parameter space, but it depends also essentially on the size of the basin of attraction for a given parameter set. Thus the question arises, how often is a particular chaotic attractor reached by trajectories started with random initial conditions compared to other attractors? Since all basins of attraction share the same state space, we have to consider the *relative size* of the basin of attraction in relation to all occurring attractors. Thus, we define the size of the basin of attraction of one attractor as the number of initial conditions which converge to that particular attractor divided by the total number of initial conditions taken into account. To compute this quantity we have chosen a grid of $[1000 \times 1000]$ points in the rectangle $[-3, 3] \times [-3, 11]$ in state space. This choice ensures that all occurring attractors (except for the one at infinity) are localized in this rectangle. We have computed the basin size of the only chaotic attractor along the line of boundary crisis points $A_{\rm cr}$ in the parameter space (A, ν). As shown in Fig. 2, we obtain a rapid decrease in the basin size of the chaotic attractor as we approach the conservative limit. Again there are vertex points in which the curve is nondifferentiable. The basin size decreases exponentially between any two successive vertex points. Both results, the computation of the existence intervals for chaotic attractors in parameter space as well as the computation of the size of the basins of attraction, yield the result that chaotic attractors in multistable systems "disappear" as we lower the damping. Qualitatively the same behavior appears for *all* attractors in the system. Towards the conservative limit, the length of the existence interval of a chaotic attractor as well as the size of its basin of attraction shrink down to zero. To obtain scaling relations about how fast the existence intervals and the basins, respectively, shrink to zero, we look at a particular path in the two-dimensional parameter space which is given by the vertices of the double crisis $v_1, v_2, \ldots$. The double crisis at v_0 does not fit to the scaling laws presented below, since at that point not all dynamics changes in the same way as for all other vertex points.

To get further insight into a double crisis vertex, let us look at one of them, say v_1, in more detail: At this point

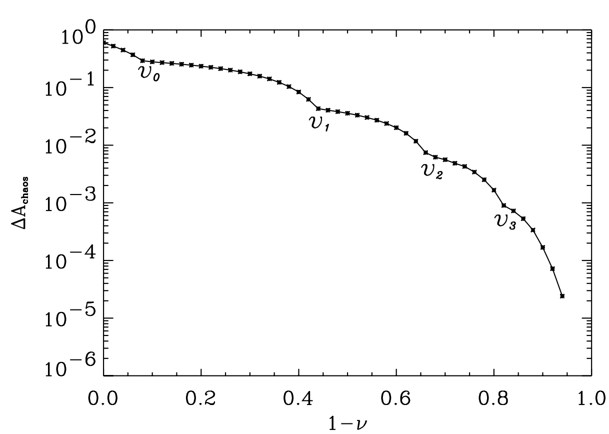

FIG. 1. Length of the chaotic interval for the Hénon map depending on the damping ν: $\Delta A_{\rm chaos}$ vs $(1-\nu)$. For a better representation we have chosen $(1-\nu)$ as the abscissa. With this choice, $(1-\nu) = 0$ corresponds to a strong damping and $(1-\nu) = 1$ corresponds to the conservative limit.

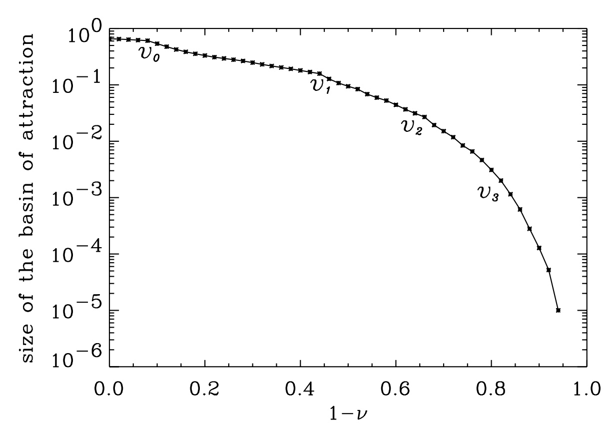

FIG. 2. Size of the basin of attraction for the chaotic attractor computed along the line of boundary crisis $A_{\rm cr}$ (right before $A_{\rm cr}$) vs $(1-\nu)$.

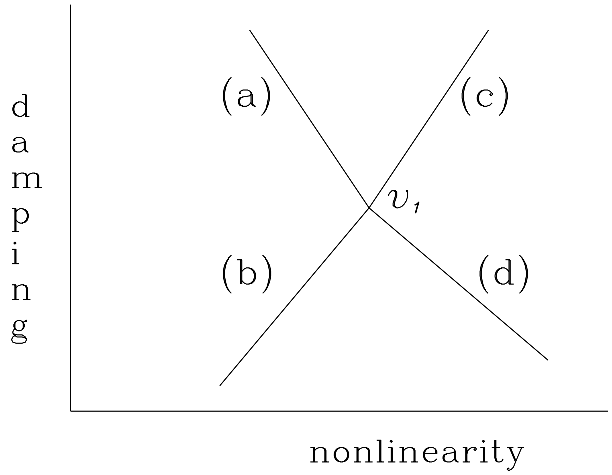

FIG. 3. Sketch of the bifurcation lines in the vicinity of a double crisis point: (*a*) interior crisis line, (*b*) basin boundary metamorphosis line, and (*c*) and (*d*) boundary crisis line.

four different bifurcation lines meet (Fig. 3). What happens dynamically as one crosses those lines?

When we cross the interior crisis line [line (*a*) in Fig. 3], there occurs a sudden change (increase or decrease depending on the direction) in the size of the attractor. In this particular case, the number of pieces of the attractor changes, above the line (*a*) there is a one-piece attractor, while below this line there are two pieces. The consequence of this interior crisis is that, along the subsequent boundary crisis lines (*c*) and (*d*), the number of attractor pieces involved in the boundary crisis changes at v_1.

Crossing the basin metamorphosis line (*b*), the fractal structure of the basin boundary changes [9]. This transition is accompanied by a change in the unstable periodic orbit located in the basin boundary that is accessible from the interior of the basin of the chaotic attractor [10]. As a consequence of the basin metamorphosis, the unstable periodic orbit with which the attractor collides at the boundary crisis changes. On the left of line (*b*), the accessible orbit on the boundary is of period 3, while on the right of line (*b*) it is of period 6. Because of the intersection of the interior crisis line (*a*) and the basin boundary metamorphosis line (*b*) at the double crisis point v_1, the two boundary crisis lines (*c*) and (*d*) exhibit a different boundary crisis. Crossing the boundary crisis line (*c*), the one-piece attractor collides with an unstable period-3 orbit in the basin boundary, while crossing the boundary crisis line (*d*) the two-piece attractor collides with an unstable period-6 orbit.

In general, as we increase k towards the conservative limit, we obtain the following changes in going from vertex v_k to vertex v_{k+1}: (i) change in the number of attractor pieces involved in boundary crisis: $2^{k-1} \to 2^k$ and (ii) change in the period of the unstable periodic orbit involved in the boundary crisis: $p \times 2^{k-1} \to p \times 2^k$, where p is the period of the unstable periodic orbit involved in the boundary crisis at v_1. Furthermore, along this special path in parameter space, going from the vertex v_k to v_{k+1}, various scaling laws describing the disappearance of chaotic attractors can be formulated.

Figure 1 verifies that the damping values at the vertex points obey the following scaling law, while approaching the conservative limit $\lim_{k\to\infty}(1 - \nu_k) = 1$ [11]:

$$1 - \nu_{k+1} = \sqrt{1 - \nu_k}, \qquad k = 1, 2, \ldots. \qquad (2)$$

On the other hand, the scaling of the length of chaotic intervals in parameter space along this path of vertices, as we approach the conservative limit $k \to \infty$, is

$$\lim_{k\to\infty} \frac{\Delta A_{\mathrm{chaos},k}}{\Delta A_{\mathrm{chaos},k+1}} = \delta_H, \qquad (3)$$

where $\delta_H = 8.72$, corresponding to the Feigenbaum constant in the conservative case.

Figure 2 shows the characteristic changes in the slope of the exponential decrease in the size of the basin of the chaotic attractor between successive vertices. Let us denote the size of the basin of attraction at vertex v_k in a bounded region by B_k. Then the slope between two vertices scales as

$$\frac{\ln B_{k+1} - \ln B_k}{(1 - \nu_{k+1}) - (1 - \nu_k)} \sim -C2^k, \qquad (4)$$

with C constant. A Taylor expansion of (2) for $\nu_k \to 0$ yields $\nu_{k+1} = \nu_k/2$ and thus $\nu_k \sim A2^{-k}$ with $A = \text{const}$. This results in $\lim_{k\to\infty} \ln(B_{k+1}/B_k) = -AC = -\text{const}$. Thus the size of the basin decreases exponentially between vertices. To illustrate this drastic drop in the basin size, we show the basins of attraction of the chaotic attractor at the first two vertices v_1 and v_2 in Figs. 4(a) and 4(b).

It is important to note that the changes in the basin size are a direct consequence of the basin metamorphosis which happens at each vertex. It could be argued that the changes in the slope at the vertex points are due to the fact that we calculate the size of the basin along the boundary crisis line, which itself already contains the vertices. To check the mechanism for the changes in the slope, we have computed the size of the basins of attraction also along the line of Feigenbaum points A_F, which is a smooth curve in parameter space with no points of nondifferentiability. The obtained curve for the size of the basins along the line A_F looks qualitatively the same as shown in Fig. 2 including the occurrence of points of nondifferentiability and changes in the slopes between them. But these points of nondifferentiability are *not identical* with the vertices in Fig. 2; they correspond to the intersections of the basin boundary metamorphosis line with the Feigenbaum line. Thus the change in the slope of the exponential decrease of the basin size is connected with the crossing of a line of basin boundary metamorphosis.

We arrived at the two scaling relations (3) and (4) numerically. However, the renormalization group approach about the accumulation point of period doublings

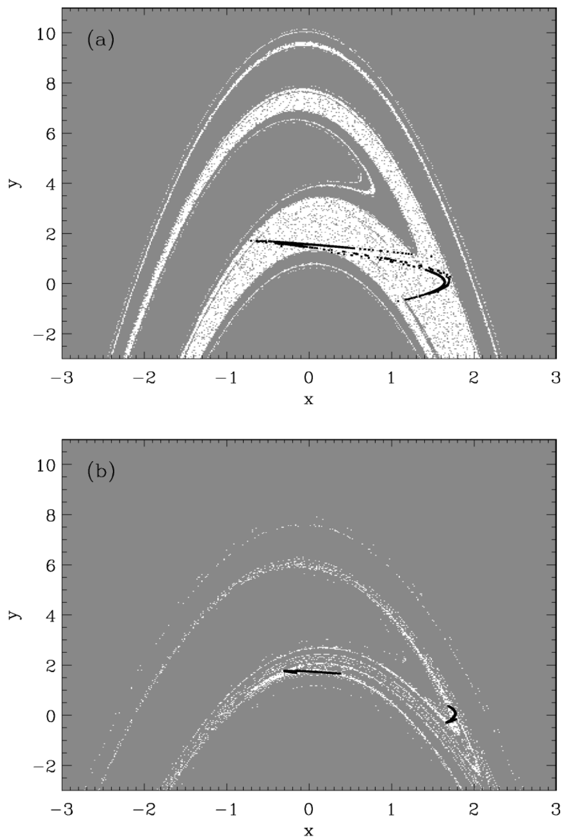

FIG. 4. Basin of attraction of the chaotic attractor in the Hénon map at two successive vertices (double crisis points): (a) at v_1 with $A_1 = 2.330\,122$, $\nu_1 = 0.557\,81$; (b) at v_2 with $A_2 = 2.926\,66$, $\nu_2 = 0.335\,026\,32$. The white dots indicate initial conditions which converge to the chaotic attractor; the grey dots indicate initial conditions which converge to other attractors. The attractor itself is shown in black.

in area-preserving two-dimensional maps yields two scaling constants from the eigenvalues λ of the renormalization transformation: one of them is $\lambda_1 = \delta_H$ for perturbations retaining the map inside the area-preserving class and the other one is $\lambda_2 = 2$ resulting from dissipation. This result is universal and holds for the class of dissipative maps possessing a period-doubling cascade and having a conservative limit [12]. The same arguments apply to the universality of the scaling relations (2) and (3), while (4) cannot be proved using a renormalization group approach due to the difficulties to quantify the size of the basins of attraction.

We conclude that chaotic attractors are not very common in weakly dissipative systems. With decreasing damping, the size of the basins of the chaotic attractors shrinks rather quickly, so that the chaotic attractors have difficulty to grow in state space. As soon as they are created, the attractors already collide with the boundary of their basin of attraction. As a consequence the extension of the chaotic region in parameter space decreases with decreasing damping. This behavior is even more pronounced, if the chaotic attractor under consideration has developed from an attractor with a higher period than 1 at the saddle-node bifurcation. Therefore, we expect chaotic attractors to be rarely observed in experimental weakly dissipative multistable systems exhibiting a multitude of attractors. Nevertheless, this does not mean that there is no chaos in such systems. Chaos shows up as long chaotic transients that are the signature for the existence of chaotic saddles which in this case are embedded in the basin boundaries of the mostly periodic attractors. In the conservative limit, the chaotic attractors in the considered system class disappear; their existence intervals as well as their basins shrink down to zero. The permanent chaos in the conservative limit of two-dimensional invertible maps appears to be connected only with the chaotic motion on chaotic saddles embedded in the basin boundaries in the dissipative case. Such scalings are also present in other diffeomorphic transformations such as the Ikeda map. How this result can be generalized to higher dimensional systems possessing Arnol'd diffusion in the conservative limit is still an open and intriguing question.

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