Lévy Model for Interstellar Scintillations

Stanislav Boldyrev[1] and Carl R. Gwinn[2]
[1]*Department of Astronomy and Astrophysics, University of Chicago, Chicago, Illinois 60637, USA*
[2]*Department of Physics, University of California, Santa Barbara, California 93106, USA*
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Observations of radio signals from distant pulsars provide a valuable tool for investigation of interstellar turbulence. The time shapes of the signals are the result of pulse broadening by the fluctuating electron density in the interstellar medium. While the scaling of the shapes with the signal frequency is well understood, the observed anomalous scaling with respect to the pulsar distance has remained a puzzle for more than 30 years. We propose a new model for interstellar electron density fluctuations, which explains the observed scaling relations. We suggest that these fluctuations obey Lévy statistics rather than Gaussian statistics, as assumed in previous treatments of interstellar scintillations.

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Introduction.—Electron density fluctuations in the interstellar medium (ISM) cause scintillations of the intensity of signals arriving from distant pulsars. If the medium were completely transparent, the shape of the arriving signal would coincide with the shape of the signal emitted by the pulsar. However, the observed pulse is much broader, and this effect is attributed to the random refraction the waves experience while they travel through the medium [1–6]. To investigate pulse broadening one can assume that the pulsar intrinsic signal is narrow in time, $I_0(t) \propto \delta(t - t_0)$, where $I_0(t)$ is the signal intensity. The observed signal is broad and asymmetric, with a sharp rise and a slow decay; see Fig. 1. Observed shapes of the pulses are similar for different pulsars (after proper rescaling), suggesting that the density fluctuation statistics along different lines of sight are to some extent universal.

For estimates assume that the pulsar distance is $d \sim 10$ kpc, the typical electron density is $n \sim 0.03$ cm^{-3}, and the observational wave frequency is $\nu \sim 500$ MHz. Then the plasma electron frequency $\omega_{pe} = (4\pi n e^2/m_e)^{1/2}$ is much smaller than ν, and density fluctuations change the wave phase only slightly. To estimate the time delay one can use the approach of geometric optics, where the propagating ray is refracted (scattered) by small prisms of density inhomogeneities [8–12]. At each scatter event occurring at a mean-free path l, the propagation angle changes by a small amount, $\Delta\theta \sim \lambda^2 r_0 \Delta n$ (see below), where λ is the wavelength, $r_0 = e^2/m_e c^2$ is the classical radius of the electron, and Δn is the density difference at characteristic separation l. Using the standard assumption that $\Delta\theta$ is random and Gaussian, one finds that the path direction deviates from a straight line by $\theta \sim \lambda^2 r_0 \Delta n_0 (d/l)^{1/2}$, where Δn_0 is the characteristic amplitude of density-difference fluctuations, and the path length deviates from the distance d by $\Delta d \sim d\theta^2 \propto \lambda^4 d^2$. The broadening time can be estimated as $\tau_d \sim \Delta d/c$, which gives the standard scaling $\tau_d \propto \lambda^4 d^2$.

Observations show that the signal width, τ_d, estimated at the half-amplitude level, scales with the wavelength according to the obtained formula, $\tau_d \propto \lambda^4$, while the scaling with distance is close to $\tau_d \propto d^4$, contradicting the analytical prediction, as is seen in Fig. 2 [15]. This paradox was first discussed by Sutton [1], and although the theory of scintillations has been developed for more than 30 years, the contradiction has resisted analytical understanding [2,3].

In this Letter we propose that the anomalous scaling with the distance is an evidence of non-Gaussian density fluctuations in the ISM. We suggest that the probability distribution of density gradients has a power-law decay, and its second moment is divergent. Such probability distributions are common in theories of turbulence, as is consistent with the argument that the density statistics are governed by turbulent motions in the ISM [17,18]. The sum of many angular deviations caused by such fluctuations does not have a Gaussian distribution; instead, the

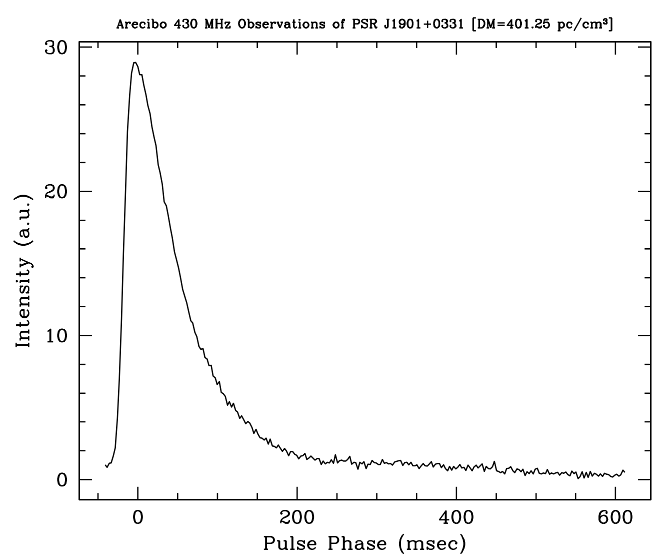

FIG. 1. Intensity of a typical observed pulsar signal averaged over many periods of pulsation. The shown time interval spans the pulsar period. The data were taken with the Arecibo telescope, at 430 MHz. (Courtesy of N. D. Ramesh Bhat [7]).

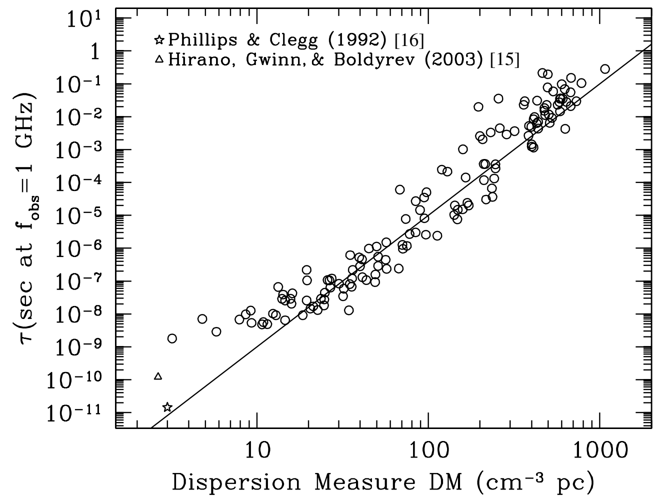

FIG. 2. Pulse temporal broadening as a function of the dispersion measure, $\mathrm{DM} = \int_0^d n(z)dz$, which is a measure of the distance to the pulsar [13]. Except as noted, data were taken from [14] see also [15,16]. The solid line has slope 4.

limiting distribution is of the Lévy type, and the ray angle performs a Lévy flight instead of a conventional random walk. We present a solvable model of scintillations that allows us to unify and extend to a non-Gaussian case the standard analytical approaches; see, e.g., [4,8]. We then apply this model to Lévy density statistics, compare it to the observational data, and demonstrate that the model naturally produces correct scalings of the signals. We report main results here; the detailed discussion is presented in [11].

Wave equation in a random medium.—The Fourier amplitude of electric (or magnetic) field, $E_\omega(\mathbf{r})$, in the isotropic ISM with dielectric permittivity ϵ_ω obeys the wave equation

$$\left[-\Delta - \frac{\omega^2}{c^2}\epsilon_\omega(\mathbf{r})\right]E_\omega(\mathbf{r}) = 0, \quad (1)$$

where $\epsilon_\omega(\mathbf{r}) = 1 - \omega_{pe}^2(\mathbf{r})/\omega^2$, and the electron plasma frequency $\omega_{pe}(\mathbf{r})$ changes slowly on the wave scale λ. Assuming that the wave propagates in the line-of-sight direction, z, we separate the quickly changing phase of the wave from the slowly changing amplitude, $E_\omega(\mathbf{r}) = \exp(iz\omega/c)\Phi_\omega(z, \mathbf{x})$, where $\mathbf{x}$ is a coordinate perpendicular to z. Substituting this into the wave equation (1), we derive the equation for the wave amplitude,

$$\left[2i\frac{\omega}{c}\frac{\partial}{\partial z} + \Delta_\perp - 4\pi r_0 n(\mathbf{x}, z)\right]\Phi_\omega(\mathbf{x}, z) = 0, \quad (2)$$

where $\Delta_\perp$ is a two-dimensional Laplacian in the $\mathbf{x}$ plane. Following [4–6], we introduce the function $I(\mathbf{r}_1, \mathbf{r}_2, t) = \Phi(\mathbf{r}_1, t)\Phi^*(\mathbf{r}_2, t)$, whose Fourier transform with respect to time is $I_\Omega(\mathbf{r}_1, \mathbf{r}_2) = \sqrt{2/\pi}\int d\omega\, \Phi_{\omega+\Omega/2}(\mathbf{r}_1)\Phi^*_{\omega-\Omega/2}(\mathbf{r}_2)$. For coinciding coordinates this function is the intensity of the radiation whose variation in time we seek. To find this function, we may first solve the equation for $V_{\omega,\Omega}(\mathbf{r}_1, \mathbf{r}_2) \equiv \Phi_{\omega+\Omega/2}(\mathbf{r}_1)\Phi^*_{\omega-\Omega/2}(\mathbf{r}_2)$, which can be derived from Eq. (2). Assuming that $\Omega \ll \omega$, we obtain

$$i\frac{\partial V}{\partial z} = \frac{2k + \Delta k}{4k^2}\frac{\partial^2 V}{\partial \mathbf{x}_2^2} - \frac{2k - \Delta k}{4k^2}\frac{\partial^2 V}{\partial \mathbf{x}_1^2} + \frac{2\pi r_0}{k}\Delta n V, \quad (3)$$

where we denoted $k = \omega/c$, $\Delta k = \Omega/c$, and $\Delta n = n(\mathbf{x}_1, z) - n(\mathbf{x}_2, z)$.

Equation (3) is hard to solve without further simplification since $n(\mathbf{x}, z)$ is an unknown random function. The standard procedure is to assume that the density fluctuations are Gaussian with a specified correlator in x and only short-scale correlations in z; see [4]. Equation (3) can then be averaged over the Gaussian ensemble of density fluctuations and over different positions in $\mathbf{x}$ space. However, the resulting solution yields a scaling of $\tau_d \propto \lambda^4 d^2$ that contradicts observations, as noted above.

We propose that the turbulent gas motions in the ISM give rise to strongly intermittent and non-Gaussian density fluctuations. If the distribution function of Δn has a power-law decay as $|\Delta n| \to \infty$ and has no second moment, then the sum of many independent ray angle deviations does not behave as a Gaussian variable (the central limit theorem does not hold). Instead, the limiting distribution, if it exists, is the Lévy distribution. A random walk whose increments are Lévy distributed is called a Lévy flight. Such processes are common in various random systems and often replace Brownian motion in turbulent systems [19].

The Fourier transform (the characteristic function) of a symmetric Lévy distribution $P_\beta(\Delta n)$ has the simple form

$$F(\mu) = \int_{-\infty}^{\infty} d\Delta n P_\beta(\Delta n)\exp(i\mu\Delta n) = \exp(-C|\mu|^\beta), \quad (4)$$

where $0 < \beta < 2$, and C is some positive constant. Equation (4) can be taken as the definition of a symmetric Lévy distribution. The sum of N Lévy distributed variables scales as $\sum^N \Delta n \sim N^{1/\beta}$, which becomes diffusion in the Gaussian limit $\beta = 2$. For $\beta < 2$, the probability distribution function has algebraic tails, $P_\beta(\Delta n) \sim |\Delta n|^{-1-\beta}$ for $|\Delta n| \to \infty$, and its second moment is divergent. We thus assume that the random density-gradient fluctuations are Lévy distributed and are short-scale correlated in z. Below we first show how Eq. (3) can be solved for a general case of non-Gaussian random density field. Following that we apply our method to Lévy distribution.

Batchelor approximation.—Propagation as described by Eq. (3) cannot be simplified in general, when Δn is not a short-scale correlated Gaussian random variable. However, analytical investigation is possible in the important case of smooth turbulent fluctuations. This case is analogous to the Batchelor limit, in the problem of turbulent random advection [20]. For that approximation in this Letter, we neglect all effects other than those of density gradients: $n(\mathbf{x}_1) - n(\mathbf{x}_2) \simeq \boldsymbol{\sigma}(z)\cdot(\mathbf{x}_1 - \mathbf{x}_2)$, where the density gradient $\boldsymbol{\sigma}(z)$ is a random variable with correlation length $l \ll d$ along z. In this approximation, the variables separate in Eq. (3) and it can be solved exactly. We leave analysis of more complicated cases for further communcation and present here results for this simple case, which captures the essential physics.

As a further simplification, consider one-dimensional variables $\mathbf{x}_1$ and $\mathbf{x}_2$. Since the variables separate, we can look for the solution in the factorized form $V(x_1, x_2, z) = U_1(x_1, z)U_2(x_2, z)$. Then the equation for U_1 reads

$$i\frac{\partial U_1}{\partial z} = -\frac{2k - \Delta k}{4k^2}\frac{\partial^2 U_1}{\partial x_1^2} + \frac{2\pi r_0}{k}\sigma(z)x_1U_1. \quad (5)$$

The analogous equation for U_2 is obtained by changing $k \rightarrow -k$. The solution of Eq. (5) is sought in the form $U_1(x_1, z) = A(z)\exp[iB(z)x_1 + iC(z)x_1^2]$, with the initial condition $U_1(z = 0) = \delta(x_1)$, if the refracting medium extends all the way up to the pulsar. Substituting this ansatz into (5), we find

$$A(z) = \frac{A_0}{\sqrt{z}}\exp\left(\frac{-i(2k - \Delta k)}{4k^2}\int_0^z B^2(z')dz'\right), \quad (6)$$

$$B(z) = \frac{-2\pi r_0}{kz}\int_0^z \sigma(z')z'dz', \qquad C(z) = \frac{k^2}{(2k - \Delta k)z}. \quad (7)$$

Note that this solution describes the path of a single ray through a sequence of density gradients $\sigma(z)$. Effects of multiple rays can be found from superposition. The intensity of received radiation can be calculated from the Fourier transform $I_\omega(z, t) = \int_{-\infty}^{\infty} V_{\omega,\Omega}(x = 0, z) \times \exp(-i\Omega t)d\Omega/\sqrt{2\pi}$. In this Fourier transform, individual ray paths will yield contributions with phase proportional to $\Omega = \Delta kc$, with coefficient equal to the travel time for that path. Cross terms describing interference of paths yield contributions that oscillate rapidly with frequency and average to zero; see, e.g., [12]. The intensity, averaged over an ensemble of statistically independent rays, is then given by the average over individual travel times or equivalently over different realizations of $\sigma(z)$.

This leads to

$$I_\omega(z, t) \propto \left\langle \delta\left(t - \frac{d}{c} - \frac{1}{2k^2c}\int_0^z B^2(z')dz'\right)\right\rangle, \quad (8)$$

where the angular brackets denote the statistical average. Note that $B(z)$ is proportional to the deflection angle θ of the ray. Formula (8) gives the shape of the signal observed at the Earth; if the scattering medium were absent, this signal would be undistorted, $I(t) \propto \delta(t - d/c)$.

Lee and Jokipii investigated Eq. (3) for short-scale correlated Gaussian density fluctuations [4]. For averaging over Gaussian $\sigma(z)$, our solution (8) reproduces those obtained by Williamson [8] with a phenomenological approach. Thus, Williamson's solutions are applicable under the assumption of smooth Gaussian density fluctuations, and when one keeps only the linear term in the expansion of Δn. Below we apply our approach to the Lévy distributed density fluctuations.

Scintillations as Lévy flights through the interstellar medium.—The averaging in formula (8) can be performed for the Lévy distributed short-scale correlated density gradients $\sigma(z)$. To do this, we represent integrals in (7) and (8) in the discretized forms; i.e., we assume that $d = nl$, $z' = ml$, and $z'' = sl$, where l is the correlation length of density fluctuations, and change $\int_0^z f(z')dz' \rightarrow \sum_{m=1}^n f(lm)l$ for an arbitrary function $f(z)$. The right-hand side of (8) is the probability distribution of the propagation time delay, $\tau = t - d/c$. For a continuous medium, this time delay is given by

$$\tau = \frac{r_0^2 l^3 \lambda^4}{8\pi^2 c}\sum_{m=1}^n\left[\frac{1}{m}\sum_{s=1}^m s\sigma_s\right]^2. \quad (9)$$

The solution (8) can now be calculated numerically as the probability distributions of this variable, under the assumption that σ_s are distributed independently, identically, and according to the Lévy law (4). We, however, need to specify the parameter β in the Lévy formula. We do this by comparing our model with observations. The observed scaling of pulse broadening is close to $\tau_d \propto \lambda^4 d^4$, while our model gives $\tau \propto \lambda^4 d^{(2+\beta)/\beta}$, as is seen from the scaling for sums of Lévy-distributed variables, $\sum^m \sigma_s \sim m^{1/\beta}$, following Eq. (4). Thus, we obtain, $\beta \approx 2/3$.

Note that standard Gaussian models of density distribution were not able to satisfy both observational scalings, λ^4 and d^4, simultaneously. Various studies of nonsmooth density fluctuations within these models have not reproduced this scaling either [10,11,21]. Our model reproduces the anomalous d scaling naturally. Moreover, it predicts that the probability distribution function of electron density gradients in the interstellar turbulence decays as $P(\sigma) \sim |\sigma|^{-1-\beta} \sim |\sigma|^{-5/3}$. Power-law distributions $P(\sigma)$ with $\beta < 2$ are indeed observed in numerical simulations of compressible turbulence [22]; however, no one has yet derived them from first principles. To date theories of scintillations have exploited only second-order correlators of the density fluctuations, while in our approach these correlators do not exist (or do not matter) and one must work with the whole probability distribution function.

Although our goal was to explain the scalings of the signals, it is interesting to see to what extent we can predict their shapes. The delay time is proportional to the square of the typical deflection angle of the ray trajectory, $\tau \propto \theta^2$, where θ has the Lévy distribution $P_\beta(\theta)$. Therefore, the distribution of arrival times is $I(\tau) \propto P_\beta(\tau^{1/2})\tau^{-1/2}$, with the asymptotic form $I(\tau) \propto \tau^{-1-\beta/2}$ as $\tau \rightarrow \infty$. Figure 3 shows the distribution of τ from numerical calculation, with a power-law decay at long times, as expected. Because the observed shape of the scattered pulse is directly related to the probability distribution of gradients in electron density in the ISM,

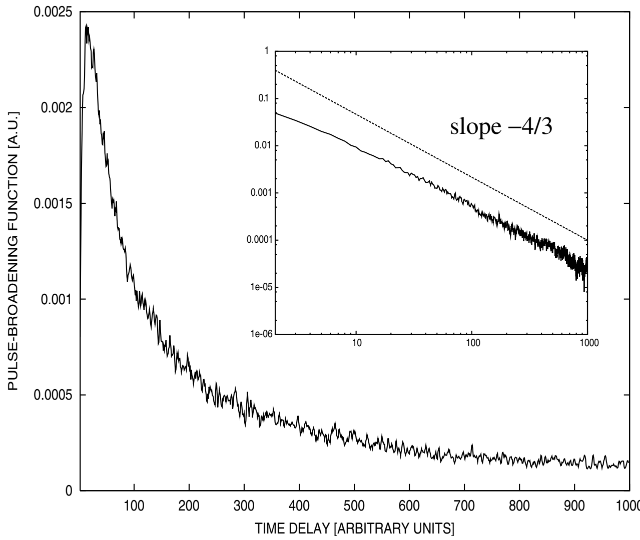

FIG. 3. Pulse-broadening functions for the model of linear density fluctuations obeying the Lévy statistics with $\beta = 2/3$. The distribution of the time delay (9) is found numerically using 10^6 rays. The inset shows the large-time asymptotics of the curve in the log-log scale.

observational data offer the possibility of characterizing interstellar turbulence [7].

The curve in Fig. 3 closely resembles the observed signals, although the presented analytical shapes are the result of averaging over an ensemble of noninterfering rays, corresponding to an observational average over an infinite amount of time. In practice, the averaging time is finite, and the long tail of the distribution, dominated by rare events, may not have converged. We also ignore instrumental response [7]. Moreover, a nonanalytic density field is more natural for a turbulent cascade [17,18,23]. Also, the small-scale density fluctuations that produce scintillations should be collisionless, and elongated along the local magnetic field [17]; the scattering is likely to be highly anisotropic, and locally nearly one dimensional. We will consider these effects in future work. Interestingly, some scattered pulsars show power-law declines at long times, such as that in Fig. 1. Scintillation of nearby pulsars also shows evidence for weak large-angle scattering [24]. Some interferometric studies suggest a "halo" surrounding the source at large scattering angles and excess scattering at small angles relative to a Gaussian [25,26], as might be expected for a Lévy distribution of scattering angles. Intrinsic source structure, and the relatively short observational averages, may complicate this interpretation.

Finally, we comment on the original explanation of the anomalous d scaling by Sutton [1]. Sutton suggested that encounters with much more strongly scattering HII regions become more probable on longer lines of sight. This, however, requires a perhaps surprisingly close coordination of DM (over 1.7 orders of magnitude) with τ_d (over 8 orders of magnitude). Sutton's proposal assumes essentially nonstationary statistics for the density distribution along z. Our proposal also invokes rare, large events, but in a statistically stationary way.

To summarize, we propose that the observed anomalously strong time broadening of pulsar signals is evidence for non-Gaussian distribution of electron density gradients in the ISM. We argue that this distribution is of the Lévy type, in accord with the turbulent origin of density fluctuations, and we present a simple model that explains the observational scalings of pulsar signals.

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