# Model for the Voltage Steps in the Breakdown of the Integer Quantum Hall Effect

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In samples used to maintain the U.S. resistance standard the breakdown of the dissipationless integer quantum Hall effect occurs as a series of dissipative voltage steps. A mechanism for this type of breakdown is proposed, based on the generation of magnetoexcitons when the quantum Hall fluid flows past an ionized impurity above a critical velocity. The calculated generation rate gives a voltage step height in good agreement with measurements on both electron and hole gases. We also compare this model to a hydrodynamic description of breakdown.

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In the integer quantum Hall effect (IQHE) regime [1], a two-dimensional electron fluid carries an almost dissipationless current and the ratio of the current $I_x$ to the Hall voltage $V_H$ is quantized in units of $e^2/h$. However, above a critical current, the dissipative voltage $V_x$ measured along the direction of current flow increases rapidly, leading to quantum Hall breakdown (QHBD). Several possible mechanisms for QHBD [2] have been proposed: avalanche heating [3], percolation due to an increase of delocalized states [4], quasielastic inter-Landau level scattering [5–7], acoustic phonon emission due to intra-Landau level scattering [8], formation of compressible metallic filaments [9], and resonant impurity scattering of electrons [10].

For certain samples, including those used to maintain the U.S. resistance standard at the National Institute of Standards and Technology (NIST), breakdown occurs as a series of up to 20 small steps in $V_x$, of roughly equal height, $\Delta V_x \simeq 5$ mV, when the applied magnetic field $B$ is increased toward the minimum in $V_x$ at 12.43 T, corresponding to Landau level (LL) filling factor $\nu = 2$; see Fig. 1 [11,12]. This shows a series of plots for 14 separate upsweeps of $B$. Steps are also observed in the $B$ downsweeps and are accompanied by hysteretic behavior. These observations are fundamentally different from those in which breakdown is observed as a single large increase of $V_x$ [3].

IQHE breakdown is not only of fundamental interest, but is also relevant to quantum metrology since a large value of $I_x$ can improve the measurement precision. In this work we develop a theoretical model to account for the dissipative steps observed by the NIST group [11,12] and others [13,14]. We show that in the presence of charged impurity-induced disorder the quantum Hall fluid (QHF) is unstable when the local fluid velocity exceeds a critical value. Under these conditions, magnetoexciton or electron-hole (e-h) pair excitations are generated spontaneously near an impurity. The voltage step height $\Delta V_x$ is directly related to the rate of formation of the pairs, which we calculate using a parameter-free model. This type of excitation of the QHF is analogous to vortex-antivortex pair formation in classical or quantum fluid flow around an obstacle [15].

Earlier work [16–18] has formulated a method for calculating the excitation dispersion relation of e-h pairs generated by exciting an electron from an occupied LL $n$ to an unoccupied LL $(n + 1)$, where $n$ is the LL index. By extending these models to include the electric field arising from a charged impurity and the Hall voltage drop across the sample, we obtain a critical electric field at which it costs no energy to generate an e-h pair at a given wave vector $Q$. We incorporate this into a calculation of the generation rate $W$ of these pairs [19,20] due to a single charged impurity. This then gives a dissipative voltage increment $\Delta V_x$.

Using the Fermi golden rule, we calculate the transition rate, $W_{n,(n+1)}$, from Landau state $(n, k_x)$ to $(n + 1, k_x')$ due to a single charged impurity in a QHF [19,20]. This gives

$$W_{n,(n+1)} = \frac{2\pi}{\hbar}\delta(\epsilon_n - \epsilon_{(n+1)}) \left| \int_r d^3r |\Xi_0(z)|^2 \phi_n(r_\perp, k_x)\phi_{(n+1)}(r_\perp, k_x')V(r) \right|^2, \quad (1)$$

where $V(r)$ is the impurity Coulomb potential, $\phi_n(r_\perp, k_x)$ is the electronic eigenfunction, in the Landau gauge, in the $x$-$y$ plane, and $\Xi_0(z)$ is the envelope function of the first electronic subband. We assume that the $n$ lower LLs are filled and the $(n + 1)$ level is empty. There is thus no static screening of the impurity charge [21]. This gives

$$W_{n,(n+1)} = \frac{2\pi}{\hbar}\delta(\epsilon_n - \epsilon_{(n+1)}) \left| \int_{q_\perp} d^2q_\perp \frac{S}{(2\pi)^2} \int_{r_\perp} d^2r_\perp \phi_n(r_\perp, k_x)\phi_{(n+1)}(r_\perp, k_x')e^{iq_\perp \cdot r_\perp} F_i(q_\perp, z_i) \right|^2, \quad (2)$$

where

$$F_i(q_\perp, z_i) = \frac{e^2}{2S\kappa q_\perp} \int_{-\infty}^{\infty} |\Xi_0(z)|^2 e^{-q_\perp |z - z_i|} dz. \quad (3)$$

  

Here $\kappa$ is the dielectric constant (for GaAs $\kappa = 0.11\ \mathrm{nF\,m^{-1}}$). If we assume that the impurity is located at the center of the subband wave function then

$$W_{n,(n+1)} = \frac{2\pi}{\hbar}\left(\frac{e^2 2\pi}{8\pi^2\kappa L_x}\right)^2 \delta(\epsilon_n - \epsilon_{(n+1)}) \left| \int_{q_y} dq_y \frac{1}{q_\perp (1+\frac{q_\perp}{b})^3} I^{k_x,k_x'}_{n,(n+1)} \right|^2, \tag{4}$$

where

$$I^{k_x,k_x'}_{n,(n+1)} = \int_y dy \Phi_n(y - Y_{k_x}) e^{iq_y y} \Phi_{(n+1)}(y - Y_{k_x'}), \tag{5}$$

$\Phi_n(y - Y_{k_x})$ is the simple harmonic oscillator solution to the Schrödinger equation centered on $Y_{k_x} = l_B^2 k_x - E_y m^\star/(eB^2)$, $m^\star$ is the effective mass of the carrier, $E_y$ is the $y$ component of the electric field, $l_B = \sqrt{\hbar/eB}$ is the magnetic length, and $B$ is the magnetic field. Calculating the transition rate out of state $(n, k_x)$ we find that

$$W_n = 2\sum_{k_x'} W_{n,(n+1)} = \frac{2L_x}{2\pi}\int_{k_x'} dk_x' W_{n,(n+1)} = \frac{1}{L_x\hbar}\left(\frac{e^2}{4\pi\kappa}\right)^2 \left|\frac{\partial k_x}{\partial \Delta\epsilon}\right|_{\Delta\epsilon=0} \left| \int_{q_y} dq_y \frac{1}{q_\perp (1+\frac{q_\perp}{b})^3} I^{k_x,k_x'}_{n,(n+1)} \right|^2, \tag{6}$$

where [16–18]

$$\Delta\epsilon = \hbar\omega_c + \frac{e^2}{4\pi\kappa l_B}\Delta_{n,(n+1)}(k_x' - k_x) - eE_y l_B^2 (k_x' - k_x) \tag{7}$$

and $\Delta_{n,(n+1)}(k_x' - k_x)$ includes the exchange and Coulomb local-field corrections, which are independent of $E_y$. To obtain Eq. (7) we have assumed $e^2/\kappa a_L \ll \hbar\omega_c$; hence we neglect LL mixing. In Eq. (6) there is an implicit condition for $k_x'$ which must be calculated. This condition can be obtained from Eq. (7) when $\Delta\epsilon = 0$. If exchange and Coulomb interactions are omitted this condition is simply the same as for energy-conserving elastic inter-Landau level transitions. However, when interaction terms are included, $Q = k_x' - k_x$ must be evaluated numerically. Equation (7) can be split into two components: the excitation interaction energy, given by the first two terms, which is independent of $E_y$ and the electrostatic energy, $eE_y l_B^2 Q$. In Fig. 2 the crossing point of the electrostatic energy (dashed line) and the excitation interaction energy (solid line) gives the value of the e-h separation ($l_B^2 Q$) for which it costs no energy to generate e-h pairs, i.e., $\Delta\epsilon = 0$. To obtain the total rate of production of e-h pairs, we take Eq. (6) and sum over all initial states such that

$$W = 2\sum_{k_x} W_n = 2\frac{L_x}{2\pi}\int_{k_x} dk_x W_n. \tag{8}$$

The above equation gives us the generation rate of e-h pairs by a single charged impurity at given $B$ and $E_y$.

Consider a local region of the sample where $E_y$ is large enough to create e-h pairs, due to scattering from a charged impurity, at a rate given by Eq. (8). Such regions can be expected to occur at high current, possibly near the sample edge where the Hall field is expected to be large [22,23]. A pair created close to an impurity will drift along the Hall bar at a velocity $v \approx E_y/B$, so one can imagine, for a fixed generation rate, a stream of e-h pairs moving along the Hall bar. Then, for $\nu = 2$ a small fraction of electrons in the lower LL ($n = 0$) have been replaced by holes, and the previously empty upper LL ($n = 1$) contains some electrons. As the e-h pairs move away from the high field region, the spacing between the electron and hole in a pair will increase and most pairs will eventually ionize by acoustic phonon emission. Because of the absence of empty states into which the excited electron can relax, and neglecting weak, second order Auger processes, we can assume

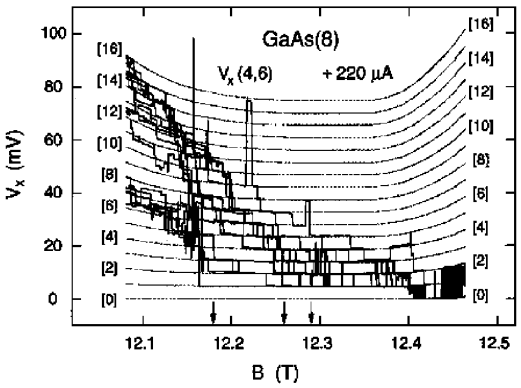


FIG. 1. Step like breakdown in the integer quantum Hall effect, from Ref. [11]. Each step contributes a multiple of ≃ 5 mV to the dissipative longitudinal voltage.

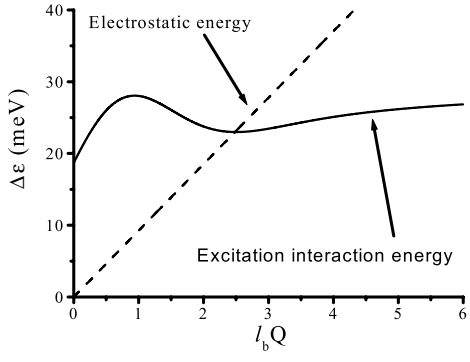


FIG. 2. The magnetoexciton mode energy, $\hbar\omega_c + e^2/(4\pi\kappa l_B)\Delta_{0,(1)}(Q)$ (solid line) for excitations from $n = 0$ to $n = 1$, for $E_y = 0$; the dashed line is the electrostatic energy $eE_y l_B^2 Q$. $B = 12.3$ T and $E_y = 1.5 \times 10^6$ V/m.

that all the generated e-h pairs will eventually ionize and lead to a dissipative current $i = eW$, flowing across the Hall voltage equipotentials. At $\nu = 2$ this gives a dissipative voltage

$$V_x = \frac{hW}{2e}. \tag{9}$$

We now compare the results of our model with breakdown measurements which show voltage steps. The NIST experiments on the U.S. resistance standard samples [11] were carried out at $\nu = 2$ and $B = 12.3$ T. The experiments show a series of dissipative steps in $V_x$ of regular height $\Delta V_x \simeq 5$ mV, see Fig. 1. In Fig. 3(a) we plot our calculated dissipative voltage as a function of the background electric field $E_y$.

Since the rate of production of e-h pairs is strongly influenced by the overlap between the wave functions in the occupied ($n = 0$) and unoccupied ($n = 1$) LLs, it increases rapidly at a critical electric field. This critical electric field occurs when Eq. (9) gives a $V_x$ comparable to the small background dissipative voltage governed by $\sigma_{xx}$. The rate, and hence $V_x$, then reaches a maximum when the electric field is such that the e-h pairs are formed close to the roton minimum of the magnetoexciton dispersion curve, $Ql_B \approx 2.5$. For a given $E_y$, for which the rate of production of e-h pairs is finite, the pairs formed at the breakdown point will relax, i.e., electrons in the upper LL relax their energy by moving toward one side of the Hall bar, while holes move in the opposite direction. This process tends to screen the Hall field over much of the Hall bar. Since the Hall voltage in the NIST experiments remains at its quantized value over the magnetic field range in which the dissipative steps occur, this screening effect tends to enhance the electric field at the breakdown point. Thus, as the critical electric field is reached, the generation rate at the breakdown point increases rapidly, inducing a further increase in $E_y$, due to the screening of the Hall field in other regions of the Hall bar. For breakdown at a single charged impurity we therefore expect the system to switch between two stable states, corresponding to $V_x = 0$ and $V_x \approx 5.6$ mV, which corresponds to the maximum value of $V_x$ in Fig. 3(a). In the NIST experiments, a series of steps is observed, as seen in Fig. 1, and we attribute each step to the formation of separate

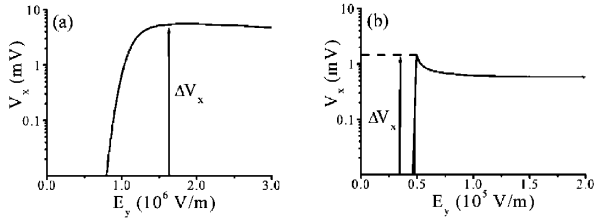


FIG. 3. The dissipative voltage for (a) electron [11] and (b) hole [15] gas samples, calculated from Eq. (9) as a function of $E_y$. The parameters used refer to the experimental conditions of [11,15]: (a) $m^\star = 0.07m_e$, $N_s = 3 \times 10^{15}$ m$^{-2}$ and $B = 12.3$ T; (b) $m^\star = 0.4m_e$, $N_s = 10^{15}$ m$^{-2}$ and $B = 2.1$ T.

streams of e-h pairs generated by other charged impurities, i.e., each step is associated with the local electric field at a particular impurity reaching its critical value.

The sharp transitions between steps and the hysteresis measured on successive up/down sweeps may be due to the redistribution of charge from the high field regions where the breakdown is initiated to the other parts of the sample. These two types of regions are known to be weakly coupled under dissipationless conditions and the instability could give rise to the sharp transition observed in the experiments.

Figure 3(b) shows the results of a similar golden rule calculation for the breakdown of the quantum Hall effect (QHE) observed in a hole gas sample [14]. In these experiments the step height was $\Delta V_x \approx 1$ mV. Our calculation for this case gives $\Delta V_x = 1.6$ mV, in qualitative agreement with experiment.

The above values are derived from a model which combines a calculation of the magnetoexciton dispersion with an impurity-related tunneling rate. We have included explicitly the mechanism for the tunneling between the LLs and exchange and Coulomb local-field corrections.

An earlier paper by one of us [15] drew an analogy between the process described here and the formation of vortices behind an obstacle moving relative to a fluid (e.g., the von Karman vortex street in classical hydrodynamics). Using our model we now examine this analogy more closely. At the mean field level [24,25], the QHF in a field corresponding to integer filling can be replaced by a system of composite bosons in zero magnetic field. Such a system forms a charged superfluid and one can view the quantum Hall state as a composite Bose condensate. From this, Stone [26] formulated an effective superfluid hydrodynamic model in which the charged elementary excitations of the QHF appear naturally as vortices in the order parameter for the composite boson superfluid. In this language, our model for the QHBD is the spontaneous creation of vortex-antivortex pairs when the QHF fluid velocity around an impurity reaches a critical value [27].

Stone's [26] equation of motion of the QHF is

$$m^\star[\dot{\mathbf{v}} - [\mathbf{v} \times \mathbf{\Omega}]] = e(\mathbf{E} + \mathbf{v} \times \mathbf{B}) - \nabla\left(\frac{m^\star}{2}|\mathbf{v}|^2 + \mu\right), \tag{10}$$

where $\mathbf{v}$ is the velocity field, $\mu$ is the local chemical potential containing all the interaction terms, and $\mathbf{\Omega}$ is the fluid vorticity. Using Eq. (10), in conjunction with the continuity equation for the density of the QHF, and examining small pertubations in the velocity field of the form

$$v_x = \frac{E_y}{B} + \epsilon_1 \cos(Qx - \omega t) \tag{11}$$

and

$$v_y = \epsilon_2 \sin(Qx - \omega t), \tag{12}$$

we find that when $\nabla\mu = 0$,

$$\hbar\omega = \Delta\epsilon = -eE_y l_B(l_B Q) + \hbar\omega_c. \quad (13)$$

This result is equivalent to Eq. (7) in the absence of interactions and for $\Delta\epsilon = 0$ corresponds exactly to the elastic inter-Landau level tunneling condition [7]. Alternatively within this hydrodynamic model it corresponds to the condition required to generate a vortex-antivortex pair at zero energy.

To make a direct comparison with our earlier quantum mechanical calculation we need to evaluate the dissipative voltage drop along the sample due to the generation of these vortex-antivortex pairs from a single impurity. For a specific system we would have to rely on a numerical simulation of Eq. (10). However, we can make considerable progress by implementing what we already know about fluid mechanics [28]. Consider an obstacle in the path of a fluid. At a low fluid velocity the flow around the obstacle is laminar. When the flow rate is increased vortex-antivortex pairs are formed in the vicinity of the obstacle. However, a vortex street is not formed until the flow is fast enough to free the vortex-antivortex pairs from the local flow field near the obstacle. In this steady state, each vortex-antivortex pair moves away from the obstacle at a velocity governed by the background fluid velocity. This analogy suggests that the vortex-antivortex pair in a QHF moves away from the impurity at a velocity given by $E_y/B$. From classical hydrodynamics [28] it is also known that the distance between each generated vortex-antivortex ($l$) pair is about 3 times the separation between a single vortex and antivortex ($d$) ($d/l = 0.28$). Now consider two states for our fluid. First, the state where the fluid flow is laminar, the generation rate of vortex-antivortex pairs is zero. For the state where we have a street of vortex-antivortex pairs we have the condition $\Delta\epsilon = 0$; hence from Eq. (13) we can evaluate $E_y/B$. Dividing this velocity by the distance between each vortex-antivortex pair [$d = 0.28 l_B(l_B Q)$], we obtain the rate of generation of vortex-antivortex pairs, and can estimate the voltage drop along the Hall bar to be

$$V_x = \frac{0.28\pi\hbar\omega_c}{e(l_B Q)^2}. \quad (14)$$

We can evaluate Eq. (14) by referring back to the previous quantum mechanical calculation to give us an estimate of $Ql_B$. Taking the value for $Ql_B$ for which the quantum calculations give the maximum value for $V_x$ we find that $\Delta V_x = 4.8$ mV for the electron gas ($Ql_B \approx 1.9$) and $\Delta V_x = 0.75$ mV for the hole gas ($Ql_B \approx 1.1$).

In summary, we have presented a quantum calculation based on magnetoexciton generation at a charge impurity to explain the presence of well-defined dissipative voltage steps measured by several groups at the onset of QHBD. We obtain a similar result using a hydrodynamical model to describe the behavior of the QHF above a critical flow velocity. The quantum model can be extended to predict the magnitude of the voltage steps for all integer filling factors and for the case of spin-dependent scattering by magnetic impurities. The fluid model can be developed to investigate the nature of the QHBD for QHF flow through more complex geometries, e.g., linear grid arrays [29] or around a single circular obstacle [30].

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