

Locally Accessible Information: How Much Can the Parties Gain by Cooperating?

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We investigate measurements of bipartite ensembles restricted to local operations and classical communication and find a universal Holevo-like upper bound on the locally accessible information. We analyze our bound and exhibit a class of states which saturate it. Finally, we link the bound to the problem of quantification of the nonlocality of the operations necessary to extract locally inaccessible information.

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The problem of local distinguishability of orthogonal quantum states has direct implications on the use of quantum correlations as a resource in quantum information theory. Although there is no clear universal delineation of what is and what is not possible for parties restricted to local operations and classical communication (LOCC), a number of interesting, often counterintuitive results have been reached. For example, *any* two pure orthogonal states can be distinguished locally as well as globally [1]. On the other hand, there are ensembles of orthogonal *product* states which cannot be locally distinguished [2,3]. Moreover, there are ensembles of locally distinguishable orthogonal states, for which one can destroy local distinguishability by *reducing* the average entanglement of the ensemble states [4].

Naturally, one would like to *quantify* the message contained in all these examples (cf. [2,5]). When there are no restrictions on the allowed measurement strategy, the classical information about the identity of the state in an ensemble $Q = \{p_x, \varrho^x\}$, accessible to a measurement, is limited by the Holevo bound [6]:

$$I_{\text{acc}} \leq \chi_Q \equiv S(Q) - \sum_x p_x S(\varrho^x), \quad (1)$$

where $\varrho = \sum_x p_x \varrho^x$ and S is von Neumann entropy. A fundamental message carried by this bound is that *the amount of information that can be sent via n qubits is bounded by n bits*. In the present contribution, we generalize bound (1) to the case when the information is coded in bipartite states and the allowed measurement strategies are limited to LOCC-based measurements. We show that for any asymptotic measure of entanglement E , *the amount of locally accessible information [7] that can be sent via n qubits with average entanglement \bar{E} is bounded by $n - \bar{E}$ bits*. We then discuss the possible saturation of our bound. Also, we link the bound to entanglement manipulations.

We begin by considering a quantum ensemble $\{p_x, \varrho_x\}$ and the mutual information $I(X:Y)$ between the signals X and the measurement results Y accessible in sequential measurements. In the first step, the measurement \mathcal{M}_1

produces outcome $a \in Y_1$ with probability p_a . In the second step, measurement \mathcal{M}_2^a [choice of \mathcal{M}_2 may depend on the result (a) of \mathcal{M}_1] gives outcomes $b_a \in Y_2$, etc. For such measurements, $I(X:Y) \equiv I(X:Y_1, Y_2, \dots)$. The well-known chain rule for mutual information (see, e.g., [8]) allows one to express the right-hand side of this identity in terms of the information gains in the single measurement steps, $I_A^1 \equiv I(X:Y_1)$, $I_B^2 \equiv I(X:Y_2|Y_1) = \sum_a p_a I(X:Y_2|Y_1 = a)$, etc. The chain rule reads

$$I(X:Y_1, Y_2, \dots) = I(X:Y_1) + I(X:Y_2|Y_1) + \dots \quad (2)$$

In other words, the total mutual information is the sum of the contributions obtained in the consecutive steps. A multistep measurement can be viewed as a tree; total mutual information is then equal to the sum of the average mutual information obtained at each level.

In addition to the chain rule, in order to proceed, we need to adapt the Holevo bound to sequential measurements, as given by the following lemma.

Lemma 1: *If a measurement on ensemble $Q = \{p_x, \varrho_x\}$ produces result y and leaves a postmeasurement ensemble $Q^y = \{p_{x|y}, \varrho_{x|y}\}$ with probability p_y , then information $I^{(1)}$ extracted from the measurement is bounded by*

$$I^{(1)} \leq \chi_Q - \bar{\chi}_Q^y, \quad (3)$$

where $\bar{\chi}_Q^y$ is the average Holevo bound for the possible postmeasurement ensembles.

To prove the lemma, we consider a system consisting of the state identifiers (X), the ensemble (Q), and the measuring device (Y).

Before the measurement, system XQY is in the state $\rho_{XQY} = \sum_x p_x |x\rangle\langle x| \otimes \rho_x \otimes |0\rangle_Y\langle 0|$. The measurement changes it to $\rho'_{XQY} = \sum_x p_x |x\rangle\langle x| \otimes V_y \rho_x V_y^\dagger \otimes |y\rangle\langle y|$ which can be rewritten as $\rho'_{XQY} = \sum_y p_y \sum_x p_{x|y} |x\rangle\langle x| \otimes \rho_x^y \otimes |y\rangle\langle y|$, where $\rho_x^y = V_y \rho_x V_y^\dagger / p_{y|x}$. We may further notice that $\bar{\chi}_Q^y = \sum_y p_y [S(\sum_x p_{x|y} \rho_x^y) - \sum_x p_{x|y} S(\rho_x^y)]$ as well as $I^{(1)} = I_M(\rho'_{X:Y})$ and $\chi_Q = I_M(\rho_{X:Q})$, where I_M is quantum mutual information, i.e., $I_M(\rho_{X:Y}) = S_X + S_Y - S_{XY}$.

Since quantum mutual information cannot increase under a local map we have

$$I_M(\rho'_{X:QY}) \leq I_M(\rho_{X:Q}) = \chi_Q. \quad (4)$$

On the other hand, the chain rule requires that $I_M(\rho'_{X:QY}) = I_M(\rho'_{X:Y}) + I_M(\rho'_{X:Q|Y})$. The first term here is $I^{(1)}$; the second is $\bar{\chi}_Q^y$, which together with (4) gives the claimed inequality (3) and the lemma.

The chain rule, together with the lemma and a little algebra, allows us to prove the following theorem.

Theorem 1: *Given ensemble $\{p_x, \rho_{AB}^x\}$ of quantum states ρ_{AB}^x on a bipartite system, the maximal mutual information $I(X:Y)$ accessible via LOCC between A and B satisfies the following inequality:*

$$I_{\text{acc}}^{\text{LOCC}} \leq S(\rho_A) + S(\rho_B) - \max_{Z=A,B} \sum_x p_x S(\rho_Z^x), \quad (5)$$

where ρ_A and ρ_B are the reductions of $\rho_{AB} = \sum_x p_x \rho_{AB}^x$, and ρ_Z^x is a reduction of ρ_{AB}^x .

In addition to the chain rule and the lemma, in order to prove the theorem we need the following two facts:

(i) Knowledge of Alice's result may reduce the entropies of Bob's parts of the ensemble states. The average reduction, $\Delta \bar{S}_B^x$ cannot, however, exceed either S_B^x (the final entropies cannot be negative) or the corresponding decrease in the entropy of Alice's parts of the states, $\Delta \bar{S}_A^x$ (a measurement on A cannot reveal more information about B than about A). Also $\Delta \bar{S}_B^x \leq \bar{S}_A^x$.

(ii) Concavity of entropy and the fact that Alice's measurement does not change Bob's density matrix $\rho_B = \sum_a p_a \rho_B^a$ (Bob's ensemble density matrix after he has learned that Alice's result was a (with probability p_a) is denoted by ρ_B^a) require that Alice's measurement does not increase entropy of Bob's part of the ensemble, i.e., $\sum_a p_a S(\rho_B^a) \leq S(\rho_B)$.

For definiteness, let Alice make the first measurement, Bob the second, Alice the third, etc. By the chain rule, the total information gain in this sequence is given by $I_{\text{acc}}^{\text{LOCC}} = \sum_{s=1} I_Z^s$ with Z denoting Alice when s is odd and Bob when s is even. The lemma bounds this as follows:

$$I_{\text{acc}}^{\text{LOCC}} \leq (\chi_A - \bar{\chi}_A^1) + (\bar{\chi}_A^2 - \bar{\chi}_A^3) + \dots \\ + (\chi_B^1 - \bar{\chi}_B^2) + (\bar{\chi}_B^3 - \bar{\chi}_B^4) + \dots \quad (6)$$

This inequality can be easily combined with the quoted fact (ii) into the following bound on information accessible in a multistep local measurement:

$$I_{\text{acc}}^{\text{LOCC}} \leq S_A + S_B - \bar{S}_A^x - \bar{S}_B^x + g_A + g_B, \quad (7)$$

where $g_B = (\bar{S}_B^x - \bar{S}_B^{x|1}) + (\bar{S}_B^{x|2} - \bar{S}_B^{x|3}) + \dots$ is the accumulated reduction of the average entropy of Bob's part of the signal states due to his knowledge of Alice's results. Likewise, $g_A = (\bar{S}_A^{x|1} - \bar{S}_A^{x|2}) + (\bar{S}_A^{x|3} - \bar{S}_A^{x|4}) + \dots$ is the accumulated reduction of the average entropy of Alice's part of the signal states due to her knowledge of Bob's results. A multiple use of fact (i) immediately implies that $g_A + g_B \leq \min(\bar{S}_A^x, \bar{S}_B^x)$, which proves the theorem. \square

While discussing the theorem, note that $S(\rho_A) + S(\rho_B) \leq n$ ($n = \log_2 d_1 d_2$ for a $d_1 \otimes d_2$ system) and $\max\{\bar{S}_A^x, \bar{S}_B^x\} \geq \bar{E}_F$, the average entanglement of formation [9] of the ensemble states. Moreover, we know that any asymptotic measure of entanglement is smaller than entanglement of formation [10]. This immediately gives the following simple bound on the locally accessible information in ensembles of bipartite states:

$$I_{\text{acc}}^{\text{LOCC}} \leq n - \bar{E} \quad (8)$$

with \bar{E} standing for any asymptotically consistent measure of the average entanglement of the ensemble states. As noted in the beginning, this formulation is a direct analog of Holevo's result. It can be seen as "entanglement correction" to Holevo bound for LOCC-based measurements. The bound (8) can also be viewed as a *complementarity relation between locally accessible information and the average bipartite entanglement*, once we write it as $I_{\text{acc}}^{\text{LOCC}} + \bar{E} \leq n$ (cf. [11]). Note also that we have here a unification of the "opposite" facts that any two Bell states are locally distinguishable and that the four Bell states are locally indistinguishable [12]. Both cases saturate relation (8). Finally, inequality (8) immediately proves that a complete orthogonal basis of multipartite states must not contain any entangled state if it is to be locally distinguishable [4] (cf. [13]).

The fact that the LOCC restriction imposed on the allowed measurements reduces Holevo bound, brings to mind associations with coarse graining. It is a well-known fact in mathematical physics that, under a smaller algebra of observables, a given state appears as having increased entropy. Likewise, if one restricts the allowed measurements to LOCC, then the restriction brings some additional entropy on the entangled states, just like that due to coarse graining. In our case, however, the set of the allowable observables (LOCC) does not have a structure of an algebra. Therefore, the additional entropy brought by the LOCC restriction cannot be easily calculated in general. Moreover, unlike Holevo's, our bound cannot be universally saturated, even in the asymptotic limit.

A possible additional source of this lack of the saturation is seen by considering an ensemble of d pure states $|i, i\rangle$, ($i = 1, \dots, d$) in $d \otimes d$. This ensemble saturates Holevo bound (the states are also locally distinguishable). On the other hand, any nontrivial measurement on Alice's side of the ensemble reduces S_B , thus making Bob's information gain g_B negative, and inequality (5) cannot be saturated. Negative g , as here, indicates an ensemble where information accessible to Alice overlaps with that to Bob. Ensembles with positive g are in a way more interesting. There, Alice's measurement not only provides valuable information about her local state, it also increases information accessible to Bob. Thus, Alice and Bob benefit from genuine cooperation while extracting information from such ensembles. A room for this cooperation permitted by LOCC allows one to, e.g., locally

distinguish any two pure orthogonal states even if these states are entangled [1]. Note that inequality (5) can be saturated only when the cooperative gain g attains its upper bound. It is then legitimate to ask about the extent to which such a situation is universal. Specifically, in $d_1 \otimes d_2$, with an *arbitrary* average entanglement $\bar{E} \leq \min\{\log_2 d_1, \log_2 d_2\}$, can one always find an ensemble with $I_{\text{acc}}^{\text{LOCC}} = n - \bar{E}$ ($n = \log_2 d_1 d_2$)?

Although we were not able to answer the question in full generality, we found an affirmative answer for $2^{n_1} \otimes 2^{n_2}$ systems, by designing a class of the required ensembles. Our ensembles are modifications of the ensemble consisting of the ‘‘canonical’’ set of mutually orthogonal maximally entangled states in $d \otimes d$ [14]. To construct a desired ensemble, we take the states $a_1|00\rangle + a_2|11\rangle$, $-a_2|00\rangle + a_1|11\rangle$, $a_1|01\rangle + a_2|10\rangle$, and $-a_2|01\rangle + a_1|10\rangle$ in $2 \otimes 2$. The ensemble consisting of these states with equal prior probabilities saturates (8) for a measurement in the computational basis. The ensemble in $2^{n_1} \otimes 2^{n_2}$ contains then (with equal prior probabilities) all the possible n_1 -times tensor products of the above four states. For example, if the partners are Alice and Bob, then in $4 \otimes 4$, the states are $(a_1|00\rangle + a_2|11\rangle)_{A_1 B_1} \otimes (a'_1|00\rangle + a'_2|11\rangle)_{A_2 B_2}$, $(a_1|00\rangle + a_2|11\rangle)_{A_1 B_1} \otimes (-a'_1|00\rangle + a'_2|11\rangle)_{A_2 B_2}$, etc., where $A_1 A_2$ is at Alice and $B_1 B_2$ is at Bob. Separate measurements by $A_1 B_1$ and by $A_2 B_2$ in their computational bases saturate bound (8) (see [15]).

Having obtained the bound $n - \bar{E}$ for $I_{\text{acc}}^{\text{LOCC}}$, one would also like to understand the deeper physical principles that facilitate it. This could, among others, help us to generalize the result into a multipartite scenario. In a search for such principles, we linked the bound to the rules of local manipulations of entanglement. This can be regarded as a step towards quantification of the nonlocality of the operations which are required to access locally inaccessible information stored in orthogonal sets of states.

Consider then an ensemble of arbitrary signal states $\{p_x, \varrho_{AB}^x\}$ and arbitrary ‘‘detector’’ states $\{\gamma_{CD}^x\}$ (cf. [4,12,16]). Initially, let the signals and the detectors be in a joint state $\varrho_{ABCD} = \sum_x p_x \varrho_{AB}^x \otimes \gamma_{CD}^x$ with relative entropy of entanglement $\mathcal{E}^{AC:BD}(\varrho_{ABCD})$ [17] in the $AC:BD$ cut. At this point, neither the signals nor the detectors are mutually orthogonal or pure. We use this setup in order to address the following question: Can the information deficit $I_{\text{acc}}^{\text{global}} - I_{\text{acc}}^{\text{LOCC}}$ be linked to the minimum potential for average entanglement production (in the distinguishing process) necessary to reach the globally accessible information $I_{\text{acc}}^{\text{global}}$? We need the setup, since gaining information about the signal (x) can destroy the signal states (ϱ_{AB}^x) and their entanglement. But the signal states can (in principle) be correlated with any ancilla (detector). An information gain about x will then usually purify the detector state, thus allowing for a potential average production of entanglement, even if the entanglement of the signal states is destroyed.

A measurement in the AB part (not necessarily restricted to LOCC) and obtaining results j with probability

q_j will leave CD in $\eta_{CD}^j = \sum_x p_x |j\rangle \gamma_{CD}^x$, thus accessing information $H_s - \sum_j q_j H(\{p_{x|j}\})$, which is no greater than $H_s - \sum_j q_j S(\eta_{CD}^j) + \sum_x p_x S(\gamma_{CD}^x)$ (equality holds for orthogonal detector states), where $H(\{r_i\}) = -\sum_i r_i \log_2 r_i$, and $H_s = H(\{p_x\})$ is the Shannon entropy of the source. Let $\delta\mathcal{E} = \bar{\mathcal{E}}_{\text{in}}^{\text{det}} - \mathcal{E}^{AC:BD}(\varrho_{ABCD})$, where $\bar{\mathcal{E}}_{\text{in}}^{\text{det}} = \sum_x p_x \mathcal{E}(\gamma^x)$. Restricting now to LOCC-based measurements in the AB part and considering it as an LOCC in the $AC:BD$ cut, we have $\delta\mathcal{E} \leq \bar{\mathcal{E}}_{\text{in}}^{\text{det}} - \sum_j q_j \mathcal{E}(\eta_{CD}^j)$, which equals $-\sum_x p_x S(\gamma^x) + \sum_j q_j S(\eta^j) - \sum_x p_x \max_{\xi^x} \text{tr} \gamma^x \log_2 \xi^x + \sum_j q_j \sum_x p_{x|j} \max_{\xi^j} \text{tr} \gamma^x \log_2 \xi^j$ (maximizations over separable states), which is in turn clearly not more than $\sum_j q_j S(\eta^j) - \sum_x p_x S(\gamma^x)$. And so we have

$$H_s - I_{\text{acc}}^{\text{LOCC}} \geq \delta\mathcal{E}. \quad (9)$$

For orthogonal ensembles, H_s is the globally accessible information ($I_{\text{acc}}^{\text{global}}$), so that $I_{\text{acc}}^{\text{global}} - I_{\text{acc}}^{\text{LOCC}} \geq \delta\mathcal{E}$, where $\delta\mathcal{E}$ is just the average amount of entanglement produced (in the distinguishing process) by a superoperator that distinguishes between the ensemble states, if we disregard the entanglement possibly left in the ensemble states. The relation (9) holds for arbitrary detectors and hence when $\delta\mathcal{E}$ is maximized over detectors. The nontrivial cases are when the orthogonal ensemble is locally indistinguishable, so that one requires a nonlocal superoperator to distinguish between them, and correspondingly one has a possibility of positive $\delta\mathcal{E}$. We assume here a ‘‘black box’’ model of the superoperator that distinguishes between the ensemble states. So, we are allowed to look at the *classical* output of the superoperator after it distinguishes, but we are *not* allowed to manipulate the *quantum* output. One may also consider the entanglement produced in the whole state in the $AC:BD$ cut and include a minimization over measurement strategies, required to (possibly nonlocally) distinguish the ensemble. This formulation is of course the same as the previous one (due to the minimization here). Thus $\delta\mathcal{E}$ gives us a notion of entanglement production, on average, in the process of (possibly nonlocally) distinguishing an ensemble when the black box is fed with the state ϱ_{ABCD} we used. We hope that it holds even when the signals and detectors are quantum correlated. That is, we conjecture that the difference between globally and locally accessible information for an ensemble of orthogonal (not necessarily pure) states is not less than the amount of the relative entropy of entanglement which can be created in a global measurement to access $I_{\text{acc}}^{\text{global}}$ (i.e., distinguish the ensemble). For nonorthogonal ensembles, there is a further reduction of globally accessible information from H_s , due to the (global) indistinguishability of nonorthogonal states, which can make the problem more complicated.

To further link relation (8) to entanglement manipulations, we prove it for a restricted case, by using the inequality in (9). Taking the orthogonal ensemble $\{p_x, \varrho_{AB}^x\}$ as $\{p_{nm}, |\psi_{nm}^{\text{max}}\rangle\}$ [14] in $d \otimes d$ and the detectors

as $\{|\psi_{nm}^{\max}\rangle^*\}$ (with conjugation in the computational basis), we have $\mathcal{E}^{AC:BD}[\mathcal{Q}_{ABCD}(\{p_{nm}\})] \leq S[\mathcal{Q}_{AC:BD}(\{p_{nm}\}) \times \mathcal{Q}_{AC:BD}(\{p_{nm} = 1/d^2\})] = 2\log_2 d - H_s$, where $S(\rho|\sigma) = \text{tr}(\rho \log_2 \rho - \rho \log_2 \sigma)$ [note that $\rho_{AC:BD}(\{p_{nm} = 1/d^2\})$ is separable (cf. [16])]. Therefore for *such ensembles*, we again obtain $I_{\text{acc}}^{\text{LOCC}} \leq 2\log_2 d - \bar{E} \equiv n - \bar{E}$ [see Eq. (9)] by a completely different method. In the general case, can we always find such detectors that $\mathcal{E}^{AC:BD}(\mathcal{Q}_{ABCD}) \leq 2\log_2 d - H_s$ or $\mathcal{E}^{AC:BD}(\mathcal{Q}_{ABCD}) \leq S_A + S_B - H_s$? We believe that understanding this question would be important in generalizing our considerations here to a multipartite scenario.

A possible way to improve the bound (5) for bipartite states would be to relate the accumulated bound on $\Delta \bar{S}_B^x$ due to Alice's measurement not to \bar{S}_A^x but to classical mutual information contained in \mathcal{Q}_{AB}^x . Likewise, there should be some room for handling nonlocality without entanglement-like examples [2–4]. For that, a possible candidate would be an object like “information deficit” [18], although with some modifications. In particular, information deficit would have to be redefined for *ensembles* rather than for states, e.g., as the information loss under a fixed map applied independently of the coming signal. Another obstacle is that the present definition of information deficit is slightly different in spirit from accessible information. In particular, it is known that to achieve $I_{\text{acc}}^{\text{LOCC}}$, one sometimes has to add pure ancillas [3], which most likely is not the case for information deficit. The proper direction would then be to extend the definition of information deficit to relative entropy loss, rather than negentropy loss.

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- [15] In $2 \otimes d$, the ensemble consisting of, with equal prior probabilities, the states $a_1|0(0 \oplus i)\rangle + a_2|1(1 \oplus i)\rangle, -a_2|0(0 \oplus i)\rangle + a_1|1(1 \oplus i)\rangle$, ($i = 0, \dots, d-1$), \oplus is addition modulo d , saturates (8). Similarly one can construct ensembles saturating (8) in $2^{n_1} \otimes (2^{n_1} + m)$ systems for all values of \bar{E} . In $3 \otimes 3$, the ensemble consisting of the states $|\psi_{i0}\rangle = a^*|0(0 \oplus i)\rangle + b^*|1(1 \oplus i)\rangle + c^*|2(2 \oplus i)\rangle$, $|\psi_{i1}\rangle = k[(b-c)|0(0 \oplus i)\rangle + (c-a)|1(1 \oplus i)\rangle + (a-b)|2(2 \oplus i)\rangle]$ and $|\psi_{i2}\rangle$, the unique state orthonormal to $|\psi_{i0}\rangle$ and $|\psi_{i1}\rangle$ in the linear span of $|0(0 \oplus i)\rangle$, $|1(1 \oplus i)\rangle$ and $|2(2 \oplus i)\rangle$, for $i = 0, 1, 2$, given with equal probabilities, where \oplus is addition modulo 3, saturates (8) for a continuous range of \bar{E} . Here $|a|^2 + |b|^2 + |c|^2 = 1$, and $k \equiv 1/\sqrt{|b-c|^2 + |c-a|^2 + |a-b|^2}$. We are willing to conjecture that such ensembles can be obtained for arbitrary systems and for all values of \bar{E} .
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