Entanglement of Formation for Symmetric Gaussian States

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We show that for a fixed amount of entanglement, two-mode squeezed states are those that maximize Einstein-Podolsky-Rosen–like correlations. We use this fact to determine the entanglement of formation for all symmetric Gaussian states corresponding to two modes. This is the first instance in which this measure has been determined for genuine continuous variable systems.

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One of the main tasks of quantum information theory is to quantify the entanglement and the quantum correlations that quantum states possess. For that, several entanglement measures have been introduced in recent years [1]. In particular, two such measures stand out for their well-defined physical meaning: the entanglement of distillation and of formation (and the corresponding asymptotic generalization, the entanglement cost) [2]. They quantify the entanglement of a state in terms of the pure state entanglement that can be distilled out of it [3] and the one that is needed to prepare it [4], respectively.

Despite a considerable effort, for the moment we can evaluate only the entanglement of formation (EOF) or the entanglement of distillation for a few sets of mixed states. The reason is that these quantities are defined [2] in terms of an optimization problem which is extremely difficult to handle analytically. Despite this fact, in a remarkable work Wootters [5] managed to derive an analytical expression for the EOF for all two-qubit states. The EOF has also been determined for highly symmetric states (isotropic states [6] and Werner states [7]). These expressions are important theoretical tools. From a more practical point of view, they can be applied to quantify the entanglement created in current experiments as well as to compare the capability of different experimental setups. For low dimensional systems without symmetries, one can still use numerical methods to determine the EOF [8], although they are often not very efficient. For infinite-dimensional systems, however, a numerical approach is not feasible.

Among all quantum states in infinite-dimensional systems, Gaussian states play an important role in quantum information. From the experimental point of view, they can be created relatively easily [9], and one can use them for quantum cryptography [10] and quantum teleportation [11,12]. On the theoretical side, separability [13] and distillability [14] criteria for bipartite systems have been fully developed. Moreover, pure Gaussian states are intimately related to Heisenberg's uncertainty relation since they minimize such a relation for position and momentum operators.

In this work we determine the EOF of all symmetric Gaussian states of two modes. Those states arise naturally in several experimental contexts: for example, when the two output beams of a parametric down converter are sent through optical fibers [9] or in atomic ensembles interacting with light [15]. In order to determine the EOF, we connect the entanglement of pure states, as measured by the von Neumann entropy of the restriction, with the type of correlations established by Einstein, Podolsky and Rosen (EPR) in their seminal paper [16]. In fact, we show that two-mode squeezed states [17] play a very special role in this relation, since they are the least entangled states for a given correlation of this type. This provides a new characterization of two-mode squeezed states. Finally, we show that the decomposition that leads to the EOF is a decomposition in terms of Gaussian states.

We consider two modes, A and B, with corresponding Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ and canonical operators $X_{A,B}$ and $P_{A,B}$. The two-mode squeezed states have the form

$$|\Psi_s(r)\rangle := \frac{1}{\cosh(r)} \sum_{N=0}^{\infty} \tanh^N(r) |N\rangle_A \otimes |N\rangle_B, \qquad (1)$$

where $r > 0$ is the squeezing parameter and $|N\rangle$ denotes the Nth Fock state, i.e., $a^\dagger a|N\rangle_A = N|N\rangle_A$, $b^\dagger b|N\rangle_B = N|N\rangle_B$, where $a = (X_A + iP_A)/\sqrt{2}$ and $b = (X_B + iP_B)/\sqrt{2}$ are annihilation operators.

In the following, we will denote by ψ an arbitrary normalized state in $\mathcal{H}$. We define its EPR uncertainty as follows:

$$\Delta(\psi) := \min\left\{1, \frac{1}{2}[\Delta^2_\psi(X_A - X_B) + \Delta^2_\psi(P_A + P_B)]\right\}, \qquad (2)$$

where, as usual, $\Delta^2_\psi(X) := \langle X\psi|X\psi\rangle - \langle\psi|X|\psi\rangle^2$, setting $\Delta^2_\psi(X) = \infty$ if ψ is not in the domain of X. Clearly, $\Delta(\psi) \in (0, 1]$. This quantity measures the degree of nonlocal correlations, and would be zero for the idealized state considered by Einstein, Podolsky, and Rosen [16]. For any state, $\Delta(\psi) < 1$ implies the existence of such nonlocal correlations. Note that this condition is met only if at least one of the uncertainties of $(X_A - X_B)/\sqrt{2}$ or $(P_A + P_B)/\sqrt{2}$ lies below 1 (the standard quantum limit). This implies that the corresponding states must possess a certain squeezing. In fact, the two-mode squeezed states (1) are standard examples of states displaying these correlations since

$$\Delta[\Psi_s(r)] = e^{-2r} < 1. \tag{3}$$

Any value of $\Delta \in (0, 1)$ is achieved by the two-mode squeezed state with squeezing parameter

$$r_\Delta := -\frac{1}{2}\ln(\Delta). \tag{4}$$

The EPR uncertainty of a given state ψ is certainly related to its entanglement. For pure states this last property is uniquely quantified by the entropy of entanglement, $E(\psi)$, which can be determined as follows. Let us write the Schmidt decomposition of ψ as [18]

$$|\psi\rangle = \sum_{N=0}^{\infty} c_N |u_N\rangle_A \otimes |v_N\rangle_B, \tag{5}$$

$\{u_N\}$ and $\{v_N\}$ are orthonormal bases in $\mathcal{H}_{A,B}$, respectively, and $c = (c_0, c_1, \ldots) \in C$, where

$$C := \{c \in l^2_{\mathbb{R}} \mid \|c\| = 1, c_N \geq c_{N+1} \geq 0 \quad \forall\, N\}.$$

Then [19]

$$E(\psi) = \mathbf{e}(c) := -\sum_{N=0}^{\infty} c_N^2 \log(c_N^2). \tag{6}$$

Note that this quantity can be infinite for some states. For the two-mode squeezed states (1) we have

$$E[\Psi_s(r)] = \cosh^2(r)\log[\cosh^2(r)] - \sinh^2(r)\log[\sinh^2(r)].$$

With the above definitions we can state the special role that two-mode squeezed states play in relation with EPR correlations and entanglement:

Proposition 1: For all $\psi \in \mathcal{H}$, $E(\psi) \geq E[\Psi_s(r_{\Delta(\psi)})]$.

In order to give a clear interpretation of this result, we reformulate it in two equivalent forms: (i) for any given $\Delta \in (0, 1)$, $E[\Psi_s(r_\Delta)] = \inf_\psi\{E(\psi)\}$ with ψ fulfilling $\Delta(\psi) = \Delta$; (ii) for any given $E \in (0, \infty)$, $\Delta[\Psi_s(r)] = \inf_\psi\{\Delta(\psi)\}$ with ψ fulfilling $E(\psi) = E$ and r such that $E[\Psi_s(r)] = E$. Note that the equivalence of this last formulation is ensured by the fact that $E[\Psi_s(r)]$ and $\Delta[\Psi_s(r)]$ are monotonically increasing and decreasing functions of r, respectively. The first statement characterizes two-mode squeezed states as the cheapest (regarding entanglement) to achieve a prescribed EPR uncertainty. The second statement characterizes two-mode squeezed states as those states with maximal EPR correlations (minimal Δ) for any given value of the entanglement.

In order to prove Proposition 1 we introduce two lemmas and the following definition. Given $c \in C$ we define

$$\boldsymbol{\delta}(c) := 1 + 2\sum_{N=0}^{\infty}(c_N^2 - c_N c_{N-1})N. \tag{7}$$

We have $\boldsymbol{\delta}(c) \leq 1$ and $\boldsymbol{\delta}(c) = \Delta(\psi)$ whenever $|u_N\rangle = |v_N\rangle = |N\rangle$ [cf. (5)].

Lemma 1: For all ψ with Schmidt decomposition (5), $\Delta(\psi) \geq \boldsymbol{\delta}(c)$.

Proof: Since $\boldsymbol{\delta}(c) \leq 1$ we can restrict ourselves to ψ with $\Delta(\psi) < 1$. Without loss of generality we can assume that $\langle\psi|a|\psi\rangle = \langle\psi|b|\psi\rangle = 0$. Otherwise we can always find ψ' fulfilling this condition, with the same Schmidt coefficients as ψ and with $\Delta(\psi') = \Delta(\psi)$ [20]. We have

$$\begin{aligned}\Delta(\psi) = 1 &+ \sum_{N=0}^{\infty} c_N^2(\langle u_N|a^\dagger a|u_N\rangle + \langle v_N|b^\dagger b|v_N\rangle) \\ &- \sum_{N,M=0}^{\infty} c_N c_M(\langle u_N|a|u_M\rangle\langle v_N|b|v_M\rangle + \text{c.c.}) \\ &\geq \min[Z(u), Z(v)],\end{aligned}$$

where

$$\begin{aligned}Z(u) := 1 &+ 2\sum_{N=0}^{\infty} c_N^2\langle u_N|a^\dagger a|u_N\rangle \\ &- 2\sum_{N,M=0}^{\infty} c_N c_M|\langle u_N|a^\dagger|u_M\rangle|^2.\end{aligned}$$

Without loss of generality let us assume that $\min[Z(u), Z(v)] = Z(u) =: Z$. We can rewrite it as $Z = \sum_{N=0}^{\infty}\sum_{M=N+1}^{\infty}(c_N - c_M)^2 X_{N,M}$, where $X_{N,M} := |\langle u_N|a^\dagger|u_M\rangle|^2 + |\langle u_M|a^\dagger|u_N\rangle|^2$. Now, since $c \in C$ we can write $(c_N - c_M)^2 \geq \sum_{R=N}^{M-1}(c_R - c_{R+1})^2$, for $M \geq N + 1$, so that

$$\begin{aligned}Z &\geq \sum_{R=0}^{\infty}(c_R - c_{R+1})^2\sum_{N=0}^{R}\sum_{M=R+1}^{\infty} X_{N,M} \\ &= \sum_{R=0}^{\infty}(c_R - c_{R+1})^2(R + 1 + 2Y_R),\end{aligned}$$

where

$$Y_R := \sum_{N=0}^{R}\left[\langle u_N|a_N^\dagger a_N|u_N\rangle - \sum_{0=M\neq N}^{R}|\langle u_N|a^\dagger|u_M\rangle|^2\right],$$

with $a_N := a - \langle u_N|a|u_N\rangle$. Now, using that $u_N \perp u_M$ for $N \neq M$ we have $\langle u_N|a^\dagger|u_M\rangle = \langle u_N|a_N^\dagger|u_M\rangle$ which, together with $\sum_{0=M\neq N}^{R}|\langle u_N|a_N^\dagger|u_M\rangle|^2 \leq \langle u_N|a_N^\dagger a_N|u_N\rangle$, yields that $Y_R \geq 0$ for all R and therefore

$$\Delta(\psi) \geq Z \geq \sum_{R=0}^{\infty}(c_R - c_{R+1})^2(R + 1) \geq \boldsymbol{\delta}(c). \qquad \square$$

Lemma 1 indicates that for a given set of Schmidt coefficients $c \in C$ EPR correlations are maximized if the Schmidt vectors are chosen to be Fock states in the right order, i.e., $|u_N\rangle = |v_N\rangle = |N\rangle$. Next we will show that for fixed Δ, the choice of Schmidt coefficients minimizing the entropy of entanglement is given by those of a two-mode squeezed state. Since the entropy and the EPR entanglement are explicitly known functionals $\mathbf{e}(c)$ and $\boldsymbol{\delta}(c)$ on the sequences $c \in C$, this is a classical constrained variational problem.

Lemma 2: For $\Delta \in (0, 1)$, and any sequence $c \in C$ with $\boldsymbol{\delta}(c) = \Delta$, we have $\mathbf{e}(c) \geq \mathbf{e}(c^\Delta) \equiv E[\Psi_s(r_\Delta)]$, where $c_N^\Delta \propto \exp(-Nr_\Delta)$ is the unique geometric sequence in C with $\boldsymbol{\delta}(c^\Delta) = \Delta$.

Sketch of proof: We apply the method of Lagrange multipliers for constrained minima to the infinitely many variables $c_0, c_1, \ldots$, leaving aside the technicalities of making this rigorous. These involve restricting c to finite dimensional spaces, then letting the dimension of the space tend to infinity, and controlling the attained minima in this limit.

With a choice of Lagrange multipliers μ and $\lambda > 0$, designed to simplify the expressions to come, we are thus looking for stationary values $c \in C$ of the functional

$$F(c, \lambda, \mu) := \mathbf{e}(c) + \frac{\lambda}{2\ln(2)}[\boldsymbol{\delta}(c) - \Delta] + \frac{(\mu + 1)}{\ln(2)}(\|c\| - 1).$$

We obtain

$$2c_N[N\lambda + \mu - \ln(c_N^2)] = \lambda[Nc_{N-1} + (N + 1)c_{N+1}], \tag{8}$$

where we have defined $c_{-1} = 1$. One can immediately see that $c_N > 0$ and thus we can divide (8) by c_N and subtract the same expression but for $N + 1$. Defining $x_N := c_{N+1}/c_N =: e^{-2r_N} \in (0, 1]$ for $N = 0, 1, \ldots$ and writing $\lambda = 2r/\sinh^2(r)$ for some $r > 0$, we find

$$x_{N+1} = x_N - A_N - B_N, \tag{9}$$

where $N = 0, 1, \ldots$ and

$$A_N = \frac{4}{N + 2}\left[\sinh^2(r_N) - \frac{r_N}{r}\sinh^2(r)\right], \tag{10a}$$

$$B_N = \frac{N}{N + 2}\left[\frac{1}{x_N} - \frac{1}{x_{N-1}}\right]. \tag{10b}$$

If we fix $r > 0$ and x_0, we have three possibilities. (i) $x_0 < e^{-2r}$. Then, by induction, x_N is decreasing, and will reach some $x_N < 0$ for finite N, which is impossible. (ii) $x_0 > e^{-2r}$. Then x_N is increasing, and the normalization condition for c cannot be fulfilled. Hence we must have the third possibility, (iii) $x_0 = e^{-2r}$, which implies that $x_N = e^{-2r}$ for all N. Hence c_N is a geometric sequence $\propto \exp(-2Nr)$. □

With the help of Lemmas 1 and 2 we are now in the position of proving Proposition 1.

Proof of Proposition 1: Given $\psi \in \mathcal{H}$, if $\Delta(\Psi) = 1$ then it is trivial. Otherwise, using (6) and Lemma 2 we have

$$E(\psi) = \mathbf{e}(c) \geq E[\Psi_s(r_{\boldsymbol{\delta}(c)})] \geq E[\Psi_s(r_{\Delta(\psi)})], \tag{11}$$

where for the last inequality we have used $r_{\Delta(\psi)} \leq r_{\boldsymbol{\delta}(c)}$ [which follows from Lemma 1 and (3)] and the fact that $E[\Psi_s(r)]$ increases monotonically with r. □

In the following, we will apply Proposition 1 to determine the EOF of symmetric Gaussian states of two modes. For a given density operator, σ, we define its covariance matrix (CM) γ as usual,

$$\gamma_{ij} := \mathrm{tr}[(R_iR_j + R_jR_i)\rho] - 2\mathrm{tr}(R_i\rho)\mathrm{tr}(R_j\rho), \tag{12}$$

where $\{R_i, i = 1, \ldots, 4\} := \{X_A, P_A, X_B, P_B\}$. Up to local unitary operations, it can always be written in the standard form [13]

$$\gamma = \begin{pmatrix} n & 0 & k_x & 0 \\ 0 & n & 0 & -k_p \\ k_x & 0 & m & 0 \\ 0 & -k_p & 0 & m \end{pmatrix}. \tag{13}$$

We will concentrate here on symmetric states, i.e., those which are invariant under exchange of subindices A and B and therefore fulfilling $m = n$. Without loss of generality we can choose $k_x \geq k_p \geq 0$. In this case, γ is a CM iff $n^2 - k_x^2 \geq 1$ and describes an entangled state iff $1 > (n - k_x)(n - k_p)$ [13]. Next we apply local (unitary) squeezing transformations to yield the state in a more appropriate form without changing its entanglement properties. In the Heisenberg picture, the transformation multiplies (divides) $X_{A,B}$ ($P_{A,B}$) by $[(n - k_p)/(n - k_x)]^{1/4}$. A simple calculation gives

$$\Delta(\sigma) = \sqrt{(n - k_x)(n - k_p)} =: \delta, \tag{14}$$

where $\Delta(\sigma)$ is defined analogously as in (2).

Our goal is to determine the EOF of σ. This is defined as $E_F(\sigma) := \inf_D \mathcal{E}(D)$, where the infimum is taken with respect to all sets of the form $D = \{p_k, \psi_k\}$ which give rise to a decomposition of σ, i.e.,

$$\sigma = \sum_k p_k|\psi_k\rangle\langle\psi_k|, \tag{15}$$

where the $\psi_k \in \mathcal{H}$ are normalized and $p_k \geq 0$. Note that the sum can run over continuous indices. For the set D we define

$$\mathcal{E}(D) := \sum_k p_k E(\psi_k). \tag{16}$$

We call the set D a decomposition of σ. A particular decomposition D_0 of σ is defined through

$$\sigma \propto \int_{\mathbb{R}^4} d\xi\, W(\xi)|\Psi_s(r_\delta)\rangle\langle\Psi_s(r_\delta)|W(\xi)^\dagger e^{-1/4\xi^T(\gamma-\gamma_\delta)^{-1}\xi},$$

where $W(\xi) = e^{i\xi^TR}$ is the Weyl displacement operator and $\gamma_\delta \leq \gamma$ is the CM of the two-mode squeezed state

(1) with squeezing parameter r_δ. Since $W(\chi)$ are local unitary operators, we have $\mathcal{E}(D_0) = E[\Psi_s(r_\delta)]$.

We also introduce the auxiliary function $f:(0,1] \rightarrow [0,\infty)$,

$$f(\Delta) = c_+(\Delta)\log[c_+(\Delta)] - c_-(\Delta)\log[c_-(\Delta)], \quad (17)$$

where $c_\pm(\Delta) := (\Delta^{-1/2} \pm \Delta^{1/2})^2/4$. One can readily show that f is a convex and decreasing function of Δ and that

$$E[\Psi_s(r_\Delta)] = f(\Delta). \quad (18)$$

Proposition 2: $E_F(\sigma) = f[\sqrt{(n-k_x)(n-k_p)}]$.

Proof: We just have to prove that for any decomposition D, $\mathcal{E}(D) \geq f(\delta)$, where δ is given in (14), since the decomposition D_0 already achieves this value, i.e., $\mathcal{E}(D_0) = E[\Psi_s(r_\delta)] = f(\delta)$ [cf. Eq. (18)]. For any decomposition we have

$$\mathcal{E}(D) \geq \sum_k p_k f[\Delta(\psi_k)] \geq f\Big[\sum_k p_k \Delta(\psi_k)\Big] \geq f(\delta).$$

The first inequality is a consequence of Proposition 1 and (18). The second inequality is due to the convexity of f. Finally, the last inequality is a consequence of the fact that $\delta \geq \sum p_k \Delta(\psi_k)$ (which can be easily checked by using the Cauchy-Schwarz inequality) together with the fact that f is a decreasing function of its argument. □

In summary, we have determined the EOF of symmetric Gaussian states by establishing a connection between EPR–like correlations and the entanglement of a state. The result implies that the measured quantities in some of the recent experiments dealing with atoms [15] and photons [21] not only qualify entanglement but also quantify it. We expect that the methods introduced here will allow one to determine the EOF and other properties of more general Gaussian states. The optimal decomposition D_0 that gives rise to the EOF is a mixture of Gaussian pure states, which means that those states are the cheapest ones in terms of entanglement to produce symmetric Gaussian states. Thus, it is tempting to conjecture that this is also true for all Gaussian states. Finally, the results presented here provide a new characterization of two-mode squeezed states as those states which achieve a maximal EPR–like correlation for a given value of the present entanglement.

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[18] Note that some of the standard properties of state transformations in finite dimensional Hilbert spaces are no longer true in infinite dimensions. For example, two states with the same Schmidt coefficients may not be transformed into each other by local unitary transformations.
[19] log is taken in base 2.
[20] We take $|\psi'\rangle = W_A \otimes W_B|\psi\rangle$, where $W_{A,B}$ are displacement (Weyl) operators fulfilling $W_A^\dagger a W_A = a - \langle\psi|a|\psi\rangle$ and $W_B^\dagger b W_B = b - \langle\psi|b|\psi\rangle$.
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