How to Measure the Spreading Width for the Decay of Superdeformed Nuclei

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A new expression for the branching ratio for the decay via the $E1$ process in the normal-deformed band of superdeformed nuclei is given within a simple two-level model. Using this expression, the spreading or tunneling width $\Gamma^\downarrow$ for superdeformed decay can be expressed entirely in terms of experimentally known quantities. We show how to determine the tunneling matrix element V from the measured value of $\Gamma^\downarrow$ and a statistical model of the energy levels. The accuracy of the two-level approximation is verified by considering the effects of the other normal-deformed states.

Since the first discovery of superdeformation in ^{152}Dy [1], one of the principal challenges has been to develop a consistent theory regarding the decay-out mechanism of the superdeformed (SD) rotational band into the normal-deformed (ND) band. Although much experimental progress has been made since this first discovery, e.g., Lauritsen *et al.* [2] and references therein, no consistent theory has been achieved for this decay-out process, and, in fact, considerable confusion still exists regarding the application of the current theoretical interpretations. The purpose of this Letter is to report on new theoretical developments, which permit a direct determination of the spreading (or tunneling) width for decay out of the SD band in nuclei in terms of experimental quantities and, thereby, to obtain the magnitude of the tunneling matrix element.

In an earlier publication [3], two of us presented a simple two-level model to explain the decay out of the SD band in nuclei. Employing a retarded Green's function approach, we obtained an exact solution for the branching ratio, F_N, for decay via the $E1$ process in the ND band, in terms of the decay widths Γ_S and Γ_N in the SD and ND potential wells, respectively; V, the tunneling matrix element connecting the SD state with the ND state, i.e., the state with which it mixes most strongly; and $\Delta = \varepsilon_N - \varepsilon_S$, the difference between the unperturbed energies of these two states (see Fig. 1 for a graphical representation of these quantities). Our result yielded

$$F_N = \frac{(1 + \Gamma_N/\Gamma_S)V^2}{\Delta^2 + \bar{\Gamma}^2(1 + 4V^2/\Gamma_N\Gamma_S)}, \tag{1}$$

where $\bar{\Gamma} = (\Gamma_S + \Gamma_N)/2$ and V is taken to be positive definite without loss of generality.

We have recently observed that Eq. (1) can also be rewritten in the form

$$F_N = \frac{\Gamma_N\Gamma^\downarrow/(\Gamma_N + \Gamma^\downarrow)}{\Gamma_S + \Gamma_N\Gamma^\downarrow/(\Gamma_N + \Gamma^\downarrow)}, \tag{2}$$

where

$$\Gamma^\downarrow = \frac{2\bar{\Gamma}V^2}{\Delta^2 + \bar{\Gamma}^2} \tag{3}$$

is the *correct* expression from Fermi's golden rule for the spreading (or tunneling) width [3]. Equation (2) clearly shows that decay into the ND band is a two-step process, and that $\Gamma^\downarrow$ is a real, physical rate, not a mere theoretical construct. Note that

$$\Gamma^\downarrow \neq 2\pi\langle V^2\rangle/D_N, \tag{4}$$

as was employed by Vigezzi *et al.* [4]. In Eq. (4), $\langle V^2\rangle$ is the mean square of the coupling matrix elements connecting the SD and ND states and D_N is the average spacing of the ND states. In fact, Eq. (4) gives the *average* spreading width over a flat distribution in Δ [3], which can deviate drastically from the exact value given by Eq. (3).

Importantly, $\Gamma^\downarrow$, as given by Eq. (3), is a measurable quantity, which can be determined from Eq. (2):

$$\Gamma^\downarrow = \frac{F_N\Gamma_N\Gamma_S}{\Gamma_N - F_N(\Gamma_N + \Gamma_S)}. \tag{5}$$

In our Table I, we have used Eq. (5) to determine $\Gamma^\downarrow$ from the values of F_N, Γ_S, and Γ_N given in Ref. [2] (Table I). Our model gives values differing by 3 to 5 orders of
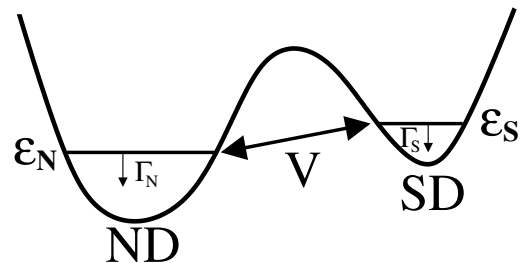

FIG. 1. Schematic diagram of the two-level problem. V is the tunneling matrix element connecting the two states. Electromagnetic decay within each band gives the states their finite widths Γ_N and Γ_S. ε_S and ε_N are the energies of the two levels in the absence of V.

TABLE I. Tunneling widths $\Gamma^{\downarrow}$ extracted from Eq. (5) compared with the results given in Table I of Ref. [2] (denoted by $\Gamma^{\downarrow(2)}$). $F_N \equiv P_{\text{out}}$, Γ_S, Γ_N, and D_N are the same as those in Table I of Ref. [2]. The spin values of the decaying states are given in parentheses.

	F_N	Γ_S	Γ_N	D_N	$\Gamma^{\downarrow}$	$\Gamma^{\downarrow(2)}$
Nucleus	$\equiv P_{\text{out}}$	(meV)	(meV)	(eV)	(meV)	(meV)
^{152}Dy(28)	0.40	10.0	17	220	11	41 000
^{152}Dy(26)	0.81	7.0	17	194	-40	220 000
^{194}Hg(12)	0.40	0.108	21	344	0.072	560
^{194}Hg(10)	0.97	0.046	20	493	1.6	37 000

magnitude from those calculated in Ref. [2] using the model of Ref. [4]. This radical difference is due to the new and physically more consistent definition of $\Gamma^{\downarrow}$ presented here.

It should be noted that we obtain a negative value for $\Gamma^{\downarrow}$ in ^{152}Dy with $I = 26$. This is, of course, physically impossible: $\Gamma^{\downarrow}$ is a positive definite quantity. Equation (5) therefore requires

$$F_N < \frac{\Gamma_N}{\Gamma_N + \Gamma_S}. \tag{6}$$

Experimental results in which the inequality (6) is violated may indicate that the experimentally measured value of F_N is too large or that the SD state is mixing with a second ND level, since our result is obtained for mixing with only one ND state. At the present time, however, uncertainties in the known values of Γ_N and Γ_S, which are of the order of 100% for Γ_N and of the order of 10% for Γ_S, or more, mean that $\Gamma^{\downarrow}$ cannot be meaningfully determined in cases such as ^{152}Dy($I = 26$).

In order to determine the tunneling matrix element V from $\Gamma^{\downarrow}$ via Eq. (3), we must know Δ, which generally is not experimentally known. We therefore compute the expected value for Δ based on the assumption that the states to which the SD state decays in the ND well are distributed according to a Gaussian orthogonal ensemble (GOE). In the GOE, the probable spacing between levels is given by the distribution [5]

$$P(s) = \frac{\pi}{2} s e^{-\pi s^2/4}, \tag{7}$$

where s is the spacing in units of D_N.

In the absence of tunneling ($V = 0$), the energy spectra of the ND and SD wells are uncorrelated. Given a spacing sD_N between the nearest ND levels above and below the decaying SD level, Δ is given by the rectangular probability distribution

$$\mathcal{P}_s(\Delta) = \frac{1}{sD_N} \Theta\left(\frac{s}{2} - \frac{|\Delta|}{D_N}\right). \tag{8}$$

Here Θ is the Heaviside step function, which simply ensures that Δ is the nearest neighbor. The probability

distribution for Δ is then

$$\mathcal{P}(\Delta) = \int_0^{\infty} \mathcal{P}_s(\Delta) P(s) ds = \frac{\pi}{2D_N} \text{erfc}\left(\sqrt{\pi}\frac{|\Delta|}{D_N}\right), \tag{9}$$

where $\text{erfc}(x)$ is the complementary error function. Figure 2 shows $\mathcal{P}(\Delta) D_N$ plotted as a function of Δ/D_N. From Eq. (9), it is easy to compute the average detuning $\langle|\Delta|\rangle = D_N/4$.

Our ultimate goal is to find the probability distribution $\mathcal{P}(V)$ for given values of $\Gamma^{\downarrow}$, $\bar{\Gamma}$, and D_N, which in general is given by $\mathcal{P}(V) = 2\mathcal{P}(\Delta)|\frac{d\Delta}{dV}|$. From Eq. (3) we can obtain $|\Delta|$ as a function of V,

$$|\Delta| = \sqrt{\frac{2\bar{\Gamma}}{\Gamma^{\downarrow}}\left(V^2 - \frac{\Gamma^{\downarrow}\bar{\Gamma}}{2}\right)}. \tag{10}$$

V obviously has a lower bound of $V_{\text{min}} = \sqrt{\frac{1}{2}\Gamma^{\downarrow}\bar{\Gamma}}$ due to the requirement that $|\Delta|$ be real.

Computing $\mathcal{P}(V)$ for the allowed region, we find

$$\mathcal{P}(V \geq V_{\text{min}}) = \frac{2\bar{\Gamma}V\pi}{\Gamma^{\downarrow}|\Delta|D_N} \text{erfc}\left(\sqrt{\pi}\frac{|\Delta|}{D_N}\right), \tag{11}$$

where $|\Delta|$ is given by Eq. (10). The average value of V is

$$\langle V \rangle = \sqrt{\frac{\Gamma^{\downarrow}}{2\bar{\Gamma}}}\left[\frac{D_N}{4} + \mathcal{O}\left(\frac{\bar{\Gamma}^2}{D_N}\right)\right]. \tag{12}$$

Equation (11) is a central result, since it represents the maximal information we can have about V without prior knowledge of the shape of the potential. Earlier attempts to consider a statistical theory of SD decay out [4,6] focused on average values of $\Gamma^{\downarrow}$ and F_N. As shown already in Ref. [4] (see also Refs. [3,7]), however, the calculated fluctuations in F_N are much larger than the mean, indicating that the average value has little meaning for comparison to experiment. Given the experimentally measured branching ratio F_N, on the other hand, our approach allows the essential parameters Δ and V to be determined, in the sense of "sharp" probability distributions whose typical values are comparable to the mean.
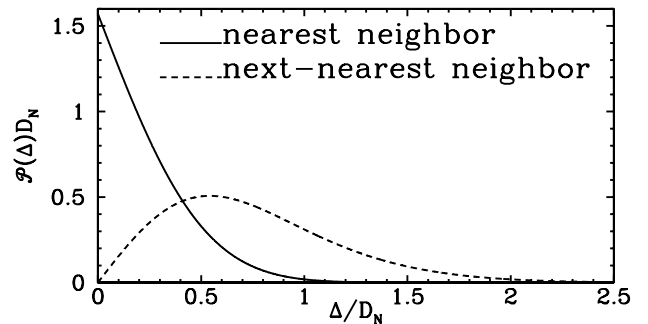

FIG. 2. Probability distributions for the two ND levels bracketing the SD level of interest, from Eqs. (9) and (14). Note that the mean nearest-neighbor spacing is $\langle|\Delta_1|\rangle = D_N/4$, while the mean spacing of the next-nearest neighbor is $\langle|\Delta_2|\rangle = 3D_N/4$.

Thus far we have treated only a two-level model of the superdeformed decay-out process. It is reasonable to ask what effect the inclusion of more ND states could have. As a first step, we include the next-nearest-neighbor state in the ND well.

We can again turn to the assumption of the GOE in order to find typical values for Δ_1 and Δ_2, the energies of the nearest- and next-nearest-neighbor states, respectively. $\mathcal{P}(\Delta_1)$ is simply the $\mathcal{P}(\Delta)$ given by Eq. (9). The calculation of $\mathcal{P}(\Delta_2)$ is similar to that for $\mathcal{P}(\Delta_1)$, but we must now concern ourselves with the question of whether Δ_1 and Δ_2 have the same or opposite signs.

In all cases of physical interest, we find that for a given spacing s, the correction to the two-state result necessitated by the inclusion of a third level is larger when the two ND levels bracket the SD level. This is quite natural, since Eq. (7) requires that the next-nearest ND level in this configuration lie on average 40% closer to the decaying SD state than it would if the nearest two ND levels lie on the same side of the SD level. Consideration of the case in which Δ_1 and Δ_2 have the same sign, then, only decreases the necessary correction. Since our goal is to set a reasonable upper bound on this quantity, we assume the ND levels lie on opposite sides of the SD level. Decays for which this is not true will in general conform to the two-state approximation with even greater accuracy.

Based on this assumption, we can construct a density function for Δ_2 similar to Eq. (8),

$$\mathcal{P}_s(\Delta_2) = \frac{1}{sD_N}\Theta\left(\frac{|\Delta_2|}{D_N} - \frac{s}{2}\right)\Theta\left(s - \frac{|\Delta_2|}{D_N}\right). \quad (13)$$

Together with Eq. (7), this yields a distribution for Δ_2,

$$\mathcal{P}(\Delta_2) = \frac{\pi}{2D_N}\left[\operatorname{erf}(\sqrt{\pi}|\Delta_2|) - \operatorname{erf}\left(\frac{\sqrt{\pi}}{2}|\Delta_2|\right)\right]. \quad (14)$$

This expression for $\mathcal{P}(\Delta_2)$ is illustrated by Fig. 2. Its average detuning is $\langle|\Delta_2|\rangle = 3D_N/4$.

Having computed the average values of Δ_1, Δ_2, and V, we are now in a position to begin to see the effect of a second ND level. In general, Eq. (1) would suggest that the contribution to the total branching ratio of a second level is substantially less than that of the nearest neighbor. Since Eq. (1) was derived in the context of only one ND level, however, we ought to seek a more rigorous theory for the three-state branching ratio. In particular, we should expect that effects such as competition and interference will play a role in the exact result.

The Hamiltonian for the three-state system can be taken to be the sum of two parts, H_0, which represents the independent SD and ND wells, and $\hat{V}$, which mixes the states of the two wells. H_0 can be written

$$H_0 = \sum_i \varepsilon_i c_i^\dagger c_i + H_{EM}, \quad (15)$$

where the sum on i runs over S, $N1$, and $N2$, c_i is the annihilation operator for state i, and H_{EM} contains the coupling to the electromagnetic field which gives the states their nonzero widths. Since they occur by the same decay process, we assume the widths of the ND states are equal, i.e., $\Gamma_{N1} = \Gamma_{N2} = \Gamma_N$.

Taking $\hat{V}$ as a perturbation, it is a trivial exercise to use Dyson's equation to construct the retarded Green's function of the system. The result, exact to all orders in $\hat{V}$, is given in the $|S\rangle$, $|N1\rangle$, $|N2\rangle$ basis by

$$G^{-1}(E) = \begin{pmatrix} E + i\frac{\Gamma_S}{2} & -V_1 & -V_2 \\ -V_1 & E - \Delta_1 + i\frac{\Gamma_N}{2} & 0 \\ -V_2 & 0 & E - \Delta_2 + i\frac{\Gamma_N}{2} \end{pmatrix}, \quad (16)$$

where V_1 and V_2 may be chosen positive without loss of generality. In the following, we assume further that $V_1 = V_2$.

The branching ratios of the full three-state system can now be computed from Parseval's theorem

$$F_i = \Gamma_i \int_{-\infty}^{\infty} \frac{dE}{2\pi} |\langle S|G(E)|i\rangle|^2, \quad (17)$$

where $i = S, N1, N2$. These integrals can be done analytically by Cauchy integration, but the results are algebraically complicated. It is sufficient for our purposes to compute them numerically.

With Δ_1 and Δ_2 determined by their probability distributions, the only remaining parameters in the three-state problem are Γ_S, Γ_N, V, and D_N. In all cases of physical interest, however, $\Gamma_S, \Gamma_N \ll V, D_N$, so we can restrict the relevant parameter space by varying only the combinations of parameters Γ_S/Γ_N and V/D_N separately. The corrections to the branching ratios required by such a restriction are of order Γ_N/V.

Furthermore, the order of Γ_S/Γ_N is determined by the mass region of the nucleus (see Table I). In the results that follow, variation of this parameter over reasonable values does not significantly impact the necessary correction to the two-state result.

Figures 3 and 4 show comparisons of the two- and three-state branching ratios for the $A \approx 190$ and $A \approx 150$ mass regions, respectively. These figures demonstrate that in cases of physical interest, the correction to the two-state system due to the presence of a third level is relatively small. In the $A \approx 190$ region, in particular, we find that $0.9 \lesssim F_N^{(2)}/F_N^{(3)} \leq 1$, where the superscripts on the branching ratios indicate the number of states included in their calculation. In the $A \approx 150$ region, we find that $0.7 < F_N^{(2)}/F_N^{(3)} < 1$. The increased importance of additional levels in the $A \approx 150$ region arises because the typical tunneling matrix elements at decay out are significantly larger, which follows from Eq. (1) and the relative sizes of Γ_S and Γ_N in the two mass regions. Constructive interference between the two ND levels,
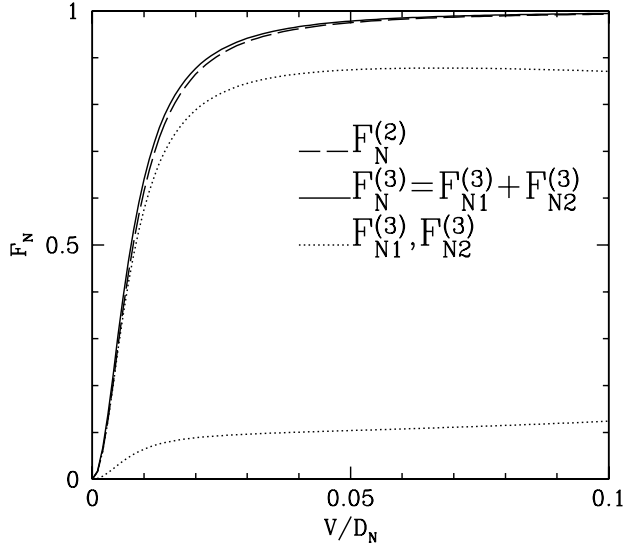
FIG. 3. Plot of ND branching ratios calculated in the two-($F_N^{(2)}$) and three-state ($F_N^{(3)}$) models with parameters relevant to the $A \approx 190$ mass region. The curves labeled $F_{N1}^{(3)}$ and $F_{N2}^{(3)}$ represent the branching ratios for the individual states of the three-level model, which sum to the total branching ratio. The energy levels were taken to lie at their mean detunings, and constant values of $\Gamma_S/\Gamma_N = 10^{-3}$ and $\bar{\Gamma}/D_N = 10^{-4}$ were used. These orders of magnitude are consistent with Table I.

FIG. 4. Plot of ND branching ratios calculated in the two- and three-state models with parameters relevant to the $A \approx 150$ mass region. The energy levels were taken to lie at their mean detunings, and constant values of $\Gamma_S = \Gamma_N$ and $\bar{\Gamma}/D_N = 10^{-4}$ were used. These orders of magnitude are consistent with Table I. The notation for branching ratios used here is defined in the caption of Fig. 3.

plainly visible in Fig. 4 where $F_N^{(3)}$ exhibits a distinct maximum for $V \sim D_N/2$, can also enhance the importance of additional ND levels.

In the 190 mass region, however, quantum interference effects are seen to be negligible, and this approach can be extended to set an approximate upper bound on the total error incurred by ignoring *all* of the ND states except the nearest neighbor. It is clear from Eq. (1) that, neglecting quantum interference effects, a level's branching ratio is approximately proportional to Δ^{-2}. Assuming that each level in the infinite ND band lies at its average detuning, we can write the total n-level branching ratio as

$$F_N^{(n)} \cong \sum_{k=1}^{n-1} \frac{F_N^{(2)}(D_N/4)^2}{(\Delta_k)^2} = \sum_{k=1}^{n-1} \frac{F_N^{(2)}(D_N/4)^2}{[(2k-1)\frac{D_N}{4}]^2}. \quad (18)$$

We thus have the result

$$F_N^{(\infty)} \cong \sum_{k=1}^{\infty} \frac{F_N^{(2)}}{(2k-1)^2} = \frac{\pi^2}{8} F_N^{(2)}. \quad (19)$$

In the 190 mass region, then, the expected correction is no more than about 23% of $F_N^{(2)}$.

The three-state results, together with the arguments of Eqs. (18) and (19), demonstrate that the two-state model is sufficient to describe the dominant decay-out process of SD nuclei. Within the two-state model, we have shown that the decay out of an SD level via $E1$ processes in the
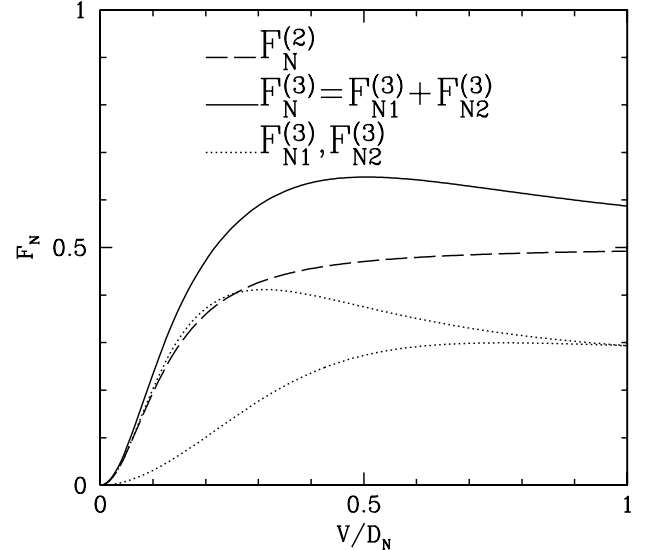
ND band is a two-step process, whose branching ratio (2) is expressed in terms of three measurable rates, Γ_S, Γ_N, and $\Gamma^{\downarrow}$. We have also shown how to determine the tunneling matrix element V [Eqs. (11) and (12)] from the measured value of $\Gamma^{\downarrow}$ and a statistical model of the ND band. It is hoped that these results will help clarify the nature of the decay-out process in SD nuclei.

[1] P. J. Twin *et al.*, Phys. Rev. Lett. **57**, 811 (1986).
[2] T. Lauritsen *et al.*, Phys. Rev. Lett. **88**, 042501 (2002).
[3] C. A. Stafford and B. R. Barrett, Phys. Rev. C **60**, 051305 (1999).
[4] E. Vigezzi, R. A. Broglia, and T. Dossing, Phys. Lett. B **249**, 163 (1990); Nucl. Phys. **A520**, 179c (1990).
[5] M. L. Mehta, *Random Matrices* (Academic Press, New York, 1991).
[6] H. A. Weidenmüller, P. von Brentano, and B. R. Barrett, Phys. Rev. Lett. **81**, 3603 (1998).
[7] J.-Z. Gu and H. A. Weidenmüller, Nucl. Phys. **A660**, 197 (1999).