An Infrared Renormalization Group Limit Cycle in QCD

Eric Braaten
Department of Physics, The Ohio State University, Columbus, Ohio 43210, USA

H.-W. Hammer
Helmholtz-Institut für Strahlen-und Kernphysik (Abteilung Theorie), Universität Bonn, 53115 Bonn, Germany
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We use effective field theories to show that small increases in the up and down quark masses would move QCD very close to the critical renormalization group trajectory for an infrared limit cycle in the three-nucleon system. We conjecture that QCD can be tuned to the critical trajectory by adjusting the quark masses independently. At the critical values of the quark masses, the binding energies of the deuteron and its spin-singlet partner would be tuned to zero and the triton would have infinitely many excited states with an accumulation point at the 3-nucleon threshold. The ratio of the binding energies of successive states would approach a universal constant that is close to 515.

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The development of the renormalization group (RG) has had a profound effect on many branches of physics. Its successes range from explaining the universality of critical phenomena in condensed matter physics to the nonperturbative formulation of quantum field theories that describe elementary particles [1]. The RG can be reduced to a set of differential equations that define a flow in the space of coupling constants. Scale-invariant behavior at long distances, as in critical phenomena, can be explained by RG flow to an infrared fixed point. Scale-invariant behavior at short distances, as in asymptotically free field theories, can be explained by RG flow to an ultraviolet fixed point. However, a fixed point is only the simplest topological feature that can be exhibited by a RG flow. Another possibility is a limit cycle, which is a one-dimensional orbit that is closed under the RG flow. The possibility of RG flow to a limit cycle was proposed by Wilson in 1971 [2]. Glazek and Wilson have recently constructed a simple discrete Hamiltonian system that exhibits a limit cycle [3]. However, few physical applications of RG limit cycles have emerged.

The purpose of this Letter is to point out that quantum chromodynamics (QCD) is close to the critical trajectory for an infrared RG limit cycle in the 3-nucleon sector. We conjecture that it can be tuned to the critical trajectory by small changes in the up and down quark masses. The proximity of the physical quark masses to these critical values explains the success of a program initiated by Efimov to describe the 3-nucleon problem in terms of zero-range forces between nucleons [4]. An effective-field-theory formulation of this program by Bedaque, Hammer, and van Kolck exhibits an ultraviolet RG limit cycle [5]. The proximity of physical QCD to the critical trajectory implies that the ultraviolet limit cycle of Ref. [5] is not just an artifact of their model.

In the late 1960s, Wilson used the RG to explain universality in critical phenomena [1]. Transformations of a system that integrate out short-distance degrees of freedom while leaving the long-distance physics invariant define a *RG flow* on the multidimensional space of coupling constants $\mathbf{g}$ for operators in the Hamiltonian:

$$\Lambda \frac{d}{d\Lambda} \mathbf{g} = \boldsymbol{\beta}(\mathbf{g}), \tag{1}$$

where Λ is an ultraviolet momentum cutoff. Standard critical phenomena are associated with *infrared fixed points* $\mathbf{g}_*$ of the RG flow, which satisfy $\boldsymbol{\beta}(\mathbf{g}_*) = 0$. The tuning of macroscopic variables to reach a critical point corresponds to the tuning of the coupling constants $\mathbf{g}$ to a *critical trajectory* that flows to the fixed point $\mathbf{g}_*$ in the infrared limit $\Lambda \to 0$. One of the signatures of an RG fixed point is *scale invariance:* symmetry with respect to the coordinate transformation $\mathbf{r} \to \lambda \mathbf{r}$ for any positive number λ. This symmetry implies that dimensionless variables scale as powers of the momentum scale, perhaps with anomalous dimensions.

In 1971, Wilson suggested that the RG might also be relevant to the strong interactions of elementary particle physics [2]. At that time, the fundamental theory for the strong interactions had not yet been discovered. It was believed to involve quarks, and hints that the strong interactions might have scaling behavior at high energies had been observed in experiments on deeply inelastic lepton-nucleon scattering. Wilson suggested that simple high-energy behavior can be explained by simple RG flow of the relevant coupling constants in the ultraviolet limit $\Lambda \to \infty$. The simplest possibility is RG flow to an *ultraviolet fixed point.* Another simple possibility is RG flow to an *ultraviolet limit cycle.* A limit cycle is a one-parameter family of coupling constants $\mathbf{g}_*(\theta)$ that is closed under the RG flow and can be parametrized by an angle $0 < \theta < 2\pi$. The RG flow carries the system around a complete orbit of the limit cycle every time the ultraviolet cutoff Λ increases by some factor λ_0. One of the signatures of an RG limit cycle is *discrete scale invariance:* symmetry with respect to

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the coordinate transformation $\mathbf{r} \to \lambda_0^n \mathbf{r}$ only for integer values of n. This symmetry implies that certain dimensionless observables, such as the ratio of the cross sections for e^+e^- annihilation into hadrons and muons, are periodic functions of the logarithm of the momentum scale with period $\ln(\lambda_0)$. The fundamental field theory for the strong interactions, QCD, was eventually discovered. QCD has a single coupling constant $\alpha_s(\Lambda)$ with an *asymptotically free* ultraviolet fixed point: $\alpha_s(\Lambda) \to 0$ as $\Lambda \to \infty$ [6].

In 1970, Efimov discovered a remarkable effect in the three-body sector for nonrelativistic particles with a resonant short-range S-wave two-body interaction [7]. The strength of the interaction is governed by the S-wave scattering length a. Efimov showed that if $|a|$ is much larger than the range r_0 of the interaction, there are shallow three-body bound states whose number increases logarithmically with $|a|/r_0$. In the resonant limit $a \to \pm\infty$, there are infinitely many shallow three-body bound states with an accumulation point at the three-body scattering threshold. If the particles are identical bosons, the ratio of the binding energies of successive states rapidly approaches the universal constant $\lambda_0^2 \approx 515$, where $\lambda_0 = e^{\pi/s_0} \approx 22.7$ and $s_0 \approx 1.006\,24$ is a transcendental number. Efimov also showed that low-energy three-body observables for different values of a are related by a discrete scaling transformation in which $a \to \lambda_0^n a$, where n is an integer, and lengths and energies are scaled by the appropriate powers of λ_0^n [7,8]. The connection between the Efimov effect and RG limit cycles was first pointed out in Ref. [9].

The Efimov effect can also occur for fermions with at least three distinct spin or isospin states. Nucleons are examples of fermions with large scattering lengths. The spin-singlet and spin-triplet np scattering lengths are $a_s = -23.8$ fm and $a_t = 5.4$ fm. They are both significantly larger than the effective range, which is $r_0 = 1.8$ fm in the spin-triplet channel. Efimov used this observation as the basis for a qualitative approach to the 3-nucleon problem [4]. He took nucleons as point particles with zero-range potentials whose strengths are adjusted to reproduce the scattering lengths a_s and a_t. The effective range and higher order terms in the low-energy expansions of the phase shifts were treated as perturbations. This approach works well in the 2-nucleon system, giving accurate predictions for the deuteron binding energy. This is no surprise; it simply reflects the well-known success of the effective-range expansion in the 2-nucleon system [10]. Remarkably, Efimov's program also works well in the 3-nucleon system at momenta small compared to m_π. In the triton channel, the Efimov effect makes it necessary to impose a boundary condition on the wave function at short distances. The boundary condition can be fixed by using the triton binding energy as input. The neutron-deuteron scattering length can then be predicted with surprising accuracy.

The Efimov effect was revisited by Bedaque, Hammer, and van Kolck (BHvK) within the framework of effective field theory (EFT) [11]. The problem of bosons with mass m and large scattering length a can be described by a nonrelativistic field theory with a complex-valued field ψ and Hamiltonian density

$$\mathcal{H} = (\hbar^2/2m)\nabla\psi^* \cdot \nabla\psi + g_2(\Lambda)(\psi^*\psi)^2. \qquad (2)$$

For convenience, we set $\hbar = 1$ in the following. In the two-body sector, the exact solution of the field theory can be obtained analytically. Renormalization can be implemented by adjusting the two-body coupling constant $g_2(\Lambda)$ as a function of the ultraviolet momentum cutoff Λ so that the scattering length is a. Other two-body observables are then independent of Λ and have the appropriate values for bosons with zero effective-range.

In the three-body sector, the nonperturbative solution of the field theory can be obtained by solving integral equations numerically. These integral equations have unique solutions only in the presence of an ultraviolet cutoff Λ. The resulting predictions for three-body observables, although finite, depend on the cutoff and are periodic functions of $\ln(\Lambda)$ with period π/s_0. BHvK showed that the quantum field theory could be fully renormalized to remove the residual dependence on Λ in the three-body sector by adding a three-body interaction term $g_3(\Lambda)(\psi^*\psi)^3$ to the Hamiltonian density in (2) [11]. The dependence of three-body observables on the cutoff decreases like $1/\Lambda^2$ if the three-body coupling constant has the form $g_3(\Lambda) = -4mg_2(\Lambda)^2 H(\Lambda)/\Lambda^2$, where

$$H(\Lambda) = \frac{\cos[s_0 \ln(\Lambda/\Lambda_*) + \arctan(s_0)]}{\cos[s_0 \ln(\Lambda/\Lambda_*) - \arctan(s_0)]} \qquad (3)$$

for some value of Λ_*. With this renormalization, three-body observables have well-defined limits as $\Lambda \to \infty$, but they depend on the parameter Λ_* introduced by dimensional transmutation. Since $H(\Lambda)$ is a periodic function of $\ln(\Lambda)$, the renormalization of the field theory involves an ultraviolet limit cycle.

An EFT of nucleons with contact interactions only has also been applied to the 3-nucleon problem [5,12]. This contact EFT involves an isospin doublet N of Pauli fields with two independent two-body contact interactions: $N^\dagger \sigma_i N^c N^{c\dagger} \sigma_i N$ and $N^\dagger \tau_k N^c N^{c\dagger} \tau_k N$, where $N^c = \sigma_2 \tau_2 N^*$. The natural scale for the ultraviolet cutoff is $\Lambda \sim m_\pi$, since the pion is the lightest degree of freedom not explicitly included in the EFT. Renormalization in the two-body sector requires that the two coupling constants be adjusted as a function of Λ to obtain the correct spin-singlet and spin-triplet scattering lengths a_s and a_t. Renormalization in the three-body sector requires the three-body contact interaction $N^\dagger \sigma_i N^c N^{c\dagger} \sigma_j N N^\dagger \sigma_i \sigma_j N$ with a coefficient proportional to (3). Thus the renormalization involves an ultraviolet limit cycle. We argue below that this ultraviolet limit cycle is a hint of an infrared limit cycle in QCD.

The low-energy few-nucleon problem can also be described by an EFT that includes explicit pion fields as well as contact interactions between the nucleons. This EFT is applicable in a domain that hopefully extends to momenta somewhat greater than m_π. The renormalization of the EFT with pions does not involve any RG limit cycle. But this has no implications for the possible existence of an infrared RG limit cycle in QCD.

Our argument is based on recent work in which an EFT with explicit pions was used to extrapolate nuclear forces to the chiral limit of QCD [13–15]. In this limit, the masses m_u and m_d of the up and down quarks are zero and the pion is an exactly massless Goldstone boson associated with spontaneous breaking of the chiral symmetry of QCD. According to these chiral extrapolations the small binding energy 2.2 MeV of the deuteron is a fortuitous consequence of the physical values of m_u and m_d. When extrapolated to the chiral limit, the deuteron has a much larger binding energy comparable to the scale $1/(m_N r_0^2) \approx 10$ MeV set by the NN effective range. Conversely, if extrapolated farther from the chiral limit, the deuteron's binding energy decreases to 0 and then it becomes unbound. This effect is illustrated in Fig. 1, which shows the chiral extrapolation of the inverse scattering lengths $1/a_t$ and $1/a_s$ as functions of m_π from Ref. [15]. In the EFT with pions, the coefficients of some of the 2-nucleon contact interactions are not constrained by data. The bands in Fig. 1 are obtained by varying those coefficients over natural ranges. The width of the error band, of course, shrinks to zero at the physical value of m_π. The prediction of Ref. [15] is that the critical value $m_{\pi,t}$ at which $1/a_t = 0$ is in the range 170 MeV $< m_{\pi,t} <$ 210 MeV, which is not much larger than the physical value of m_π. The extrapolation of $1/a_s$ has larger uncertainties. It may increase to zero at some critical value $m_{\pi,s}$ greater than 150 MeV, in which case the spin-singlet deuteron is bound for $m_\pi > m_{\pi,s}$, or it may remain negative, in which case the spin-singlet deuteron remains unbound. Beane and Savage [14] have argued that the extrapolation errors in the chiral limit are larger than estimated in Ref. [15], but their errors agree for the small extrapolation to the region of larger m_π where the deuteron becomes unbound.

We now consider the chiral extrapolation of the three-body spectrum. This could be calculated using an EFT with explicit pions. Alternatively, the chiral extrapolation can be calculated using the contact EFT of Ref. [5]. The inputs required are a_s, a_t, and Λ_* as functions of m_π, which can be calculated using an EFT with pions. For the inverse scattering lengths $1/a_s$ and $1/a_t$, we take the central values of the error bands obtained from the chiral extrapolation in Ref. [15]. The dependence of Λ_* on m_π could be determined from the chiral extrapolation of the triton binding energy using an EFT with pions, but this has not yet been calculated. Like the inverse scattering lengths, Λ_* should vary smoothly with m_π. For small extrapolations of m_π from its physical value, we can approximate Λ_* by a constant. We use the value $\Lambda_* = 189$ MeV for $m_\pi = 138$ MeV obtained by taking the triton binding energy as the three-body input. In Fig. 2, we show the three-body spectrum in the triton channel as a function of m_π. The crosses give the binding momenta $\kappa = (mB_3)^{1/2}$ of the physical deuteron and triton, while the dashed lines give the thresholds for decay into a nucleon plus a deuteron (left curve) or a spin-singlet deuteron (right curve) in the large-a approximation. The circles indicate the triton ground state and excited state. In the region near $m_\pi \approx 175$ MeV where the decay threshold comes closest to the 3-nucleon threshold $\kappa = 0$, an excited state of the triton appears. The existence of this very shallow state is a hint that the

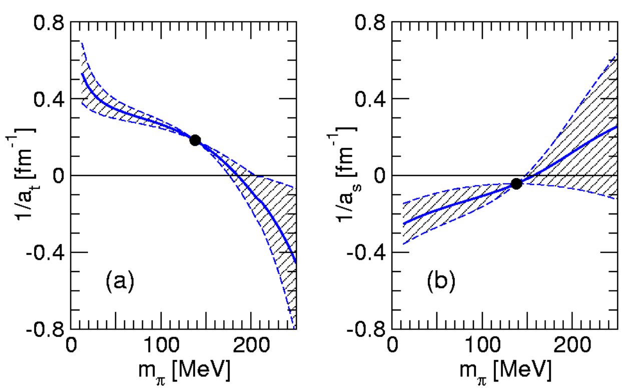

FIG. 1 (color online). The inverse scattering lengths $1/a_t$ and $1/a_s$ as functions of m_π as predicted by the EFT with pions of Ref. [15].

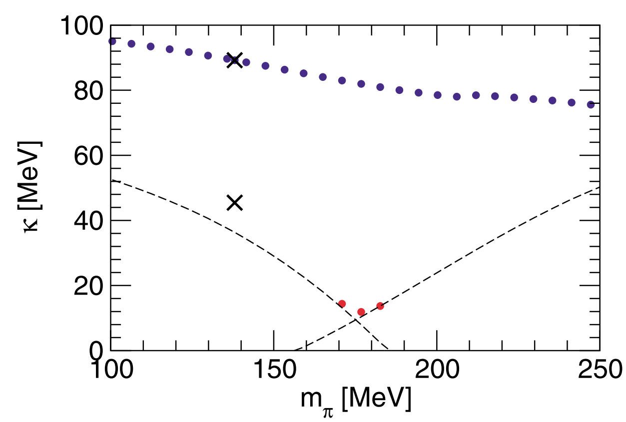

FIG. 2 (color online). The binding momenta $\kappa = (mB_3)^{1/2}$ of pnn bound states as a function of m_π calculated using the contact EFT of Ref. [5]. The circles indicate the triton ground state and excited state. The crosses give the binding energy of the physical deuteron and triton, while the dashed lines give the thresholds for decay into a nucleon plus a deuteron (left curve) or a spin-singlet deuteron (right curve).

system is very close to the critical RG trajectory for an infrared limit cycle.

If, as in the case illustrated by Fig. 2, the values $m_{\pi,t}$ and $m_{\pi,s}$ at which $1/a_t$ and $1/a_s$ go to zero satisfy $m_{\pi,s} < m_{\pi,t}$, then the deuteron and spin-singlet deuteron are both bound in the region $m_{\pi,s} < m_\pi < m_{\pi,t}$. Since the decay threshold never extends all the way down to the 3-nucleon threshold $\kappa = 0$, there cannot be an infrared limit cycle. The error bands in Ref. [15] do not exclude $m_{\pi,t} < m_{\pi,s}$, in which case the deuteron and spin-singlet deuteron are both unbound in the region $m_{\pi,t} < m_\pi < m_{\pi,s}$. The decay threshold then extends all the way down to $\kappa = 0$, but there is still no infrared limit cycle, because any finite scattering length will act as an infrared cutoff in the three-body sector.

We conjecture that QCD can be tuned to the critical RG trajectory for an infrared limit cycle by adjusting the up and down quark masses m_u and m_d. In the next-to-leading order chiral extrapolation of Ref. [15], only quark-mass dependent operators proportional to $m_u + m_d$ enter. The extrapolation in m_π can be interpreted as an extrapolation in the sum $m_u + m_d$, with $m_u - m_d$ held fixed. Changing $m_u - m_d$ changes the degree of isospin-symmetry breaking. Since a_s and a_t correspond to different isospin channels, they respond differently to changes in $m_u - m_d$. Therefore it may be possible by tuning both m_u and m_d to make $1/a_t$ and $1/a_s$ vanish simultaneously: $m_{\pi,t} = m_{\pi,s}$. This point corresponds to a critical RG trajectory for an infrared limit cycle. At this critical point, the triton has infinitely many increasingly shallow excited states with an accumulation point at the 3-nucleon threshold. The ratio of the binding energies of successively shallower states rapidly approaches a constant λ_0^2 close to 515. Now consider a RG transformation that integrates out energies above some scale Λ^2/m_N. As Λ is decreased, the deepest 3-nucleon bound states are removed from the spectrum, leaving only those for which the deviations from the asymptotic ratio λ_0^2 are negligible. Thus a limit cycle with a discrete-scaling-symmetry factor λ_0 is approached in the infrared limit $\Lambda \to 0$. Note that one can infer the existence of the infrared limit cycle from the discrete scaling symmetry without constructing the RG flow explicitly.

The proximity of physical QCD to the critical trajectory for the infrared limit cycle explains the success of Efimov's program [4] for describing the 3-nucleon problem using zero-range forces or a contact EFT [5]. The apparent convergence of the effective-range corrections for momenta of order m_π [12] could be explained if the momentum expansion in the contact EFT is in powers of p/m_π^*, where $m_\pi^* \approx 175$ MeV is the critical value of the pion mass.

The proximity of physical QCD to the critical trajectory for the infrared limit cycle has implications for efforts to derive nuclear physics from lattice gauge theory. The computational effort for lattice simulations increases dramatically as the pion mass decreases and is prohibitive at the physical value. Lattice simulations are typically carried out at a value of m_π that is 2–3 times larger than the physical value, and then a chiral extrapolation is made to $m_\pi = 138$ MeV. This requires extrapolating past the region of m_π where there is an RG limit cycle, which may introduce large extrapolation errors in three-body observables.

Lattice gauge calculations can help test our conjecture that QCD can be tuned to the critical trajectory for the infrared limit cycle by adjusting the quark masses. Because the scattering lengths a_s and a_t are so large, direct calculations of few-nucleon observables are currently not possible at the physical values of the quark masses, and even more difficult closer to their critical values. Such calculation may, however, be feasible at larger values of the quark masses where a_s and a_t have natural values of order $1/m_\pi$. If this region overlaps with the domain of validity of the EFT with pions, lattice calculations can be used to constrain the quark-mass dependence of the low-energy constants in the EFT. One might then be able to use a combination of lattice QCD and EFT to demonstrate the existence of the infrared RG limit cycle of QCD.

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