Relativistic Linear Response Wave Functions and Dynamic Scattering Tensor for the $ns_{1/2}$ States in Hydrogenlike Atoms

V. Yakhontov*

Institut für Physikalische Chemie, Universität Basel, Klingelbergstrasse 80, 4056 Basel, Switzerland

(Received 23 April 2003; published 27 August 2003)

We report a novel closed-form analytic representation for the linear response relativistic wave function of the hydrogenic $ns_{1/2}$ level that is exposed to dipole radiation of frequency ω. This result is derived by means of a direct analytical solution of the inhomogeneous ω-dependent Dirac equation. The utility of the formulas obtained is demonstrated by new analytic and numerical calculations of the static and dynamic relativistic dynamic polarizabilities of the lowest hydrogenic $ns_{1/2}$ states.

DOI: 10.1103/PhysRevLett.91.093001 PACS numbers: 31.30.Jv

Ab initio studies of laser-induced processes with few-particle quantum systems have attracted considerable attention over the last few decades [1]. Currently, this activity is revitalized and has acquired additional interest due to recent highly precise spectroscopic measurements on fundamental bound systems (e.g., see [2] and references therein), such as hydrogen, positronium, muonium, helium, H_2^+ and D_2^+ ions. In the course of these studies, respective systems are usually exposed to a tunable laser radiation, pulsed or continuous. This enables resonant multiphoton bound-bound and/or bound-free transitions to be induced and analyzed, while taking advantage of ultrahigh resolution experimental methods. At the same time, spurious field-induced perturbations, which make their appearance even at rather weak laser signals, have to be allowed for and removed from the spectra observed by carrying out elaborate simulations of the multiphoton dynamics [3,4] which such systems undergo in the presence of external fields. An extremely high precision inherent in the above measurements makes data obtained thereby sensitive enough not only to most types of conventional corrections, but also to relativistic, radiative and QED effects as well [5]. As a result, this urges a fully relativistic reformulation of a wide class of field-related atomic and molecular characteristics, the most crucial being the relativistic dynamic tensor of linear scattering (RDTLS). For any pair of eigenstates, $|i\rangle$ and $|f\rangle$, of a hydrogenic ion with a nuclear charge Z and reduced mass $m^* = Mm/(M+m)$, this fundamental second order quantity can be expressed as (the reduced atomic units, $e^2 = \hbar = m^* = 1$, are used throughout)

$$[C_{kl}(\omega, \omega')]_{fi} = -\langle f|\hat{Q}^{(2)}_{kl}(\omega, \omega')|i\rangle, \tag{1}$$

where

$$\hat{Q}^{(2)}_{kl}(\omega, \omega') \equiv \frac{c^2}{\omega\omega'}\Big[\alpha_k(\varepsilon_i - \omega - \hat{H})^{-1}\alpha_l + \alpha_l(\varepsilon_i + \omega' - \hat{H} + i0)^{-1}\alpha_k\Big]. \tag{2}$$

Here $\hat{H} = -ic\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + c^2\beta - (Z/r)$ is the standard relativistic Hamiltonian of an isolated hydrogenic atom with ε_i, ε_f being its eigenvalues; the subscripts k, l denote Cartesian components of the Dirac matrix $\boldsymbol{\alpha}$ and the tensor operator $\hat{Q}^{(2)}(\omega, \omega')$; ω, ω' are arbitrary photon energies and $c \approx 137$ is the speed of light.

Despite its considerable age, there have been quite a few attempts to find a practically useful general analytical representation for $[C_{kl}(\omega, \omega')]_{fi}$ of Eq. (1). The majority of accurate RDTLS calculations [6–10], mostly pure numerical, have so far been carried out for the static $1s_{1/2} - 1s_{1/2}$ or $1s_{1/2} - 2s_{1/2}$ transitions in hydrogenic systems. This is in contrast with the nonrelativistic DTLS studies, wherein the closed form of the Coulomb Green's function has enabled exact analytic results for any states $|f\rangle, |i\rangle$, and energies ω, ω' to be obtained [11]. On the other hand, the use of the relativistic Coulomb Green's function in analytic RDTLS studies is usually fraught with considerable technical difficulties, thus often requiring alternative analytic techniques [12] and/or pure numerical methods [7,10,13] to be employed instead. Even for the ground state RDTLS, currently available analytical results [8,9,12] do not admit straightforward numerical evaluation at $\omega, \omega' > I_{1s}$. This limitation prevents the appropriate formulas from being used directly in the above simulations, which the single photoionization of levels may enter along with other open channels of their excitations.

In this work $[C_{kl}(\omega, \omega')]_{fi}$ is found analytically and exactly by making use of relativistic version of a method due to Podolsky [14], Sternheimer [15], Dalgarno, and Lewis [16]. The latter will hereafter be referred to as the PSDL method and constitutes direct solving an equation satisfied by the linear (i.e., first-order) response relativistic wave function (LRRWF) of the state $|i\rangle$,

$$\boldsymbol{\Psi}_i(E) \equiv c(E - \hat{H})^{-1}\boldsymbol{\alpha}\,|\,i\rangle. \tag{3}$$

Thus $\boldsymbol{\Psi}_i(E)$ can be considered as the result of the convolution of the one-particle relativistic Green's function, $\hat{G}_E = (E - \hat{H})^{-1}$, with $c\boldsymbol{\alpha}|i\rangle$; the subscript i on $\boldsymbol{\Psi}_i(E)$ will be dropped henceforth to simplify notations. Even though there usually exists a formidable problem of casting expression for $\boldsymbol{\Psi}(E)$ into a practically useful analytic form, relativistic calculations based upon the PSDL

approach appear to be technically simpler, particularly numerical ones, as compared to those using the Green's function techniques. This advantage has been pointed out in the former nonrelativistic PSDL studies (see [17] and references therein), both numerical and analytic.

In the spirit of the PSDL method, we act with $E - \hat{H}$ on both sides of Eq. (3) to give

$$(E - \hat{H})\Psi_\mu(E) = c\,\alpha_\mu|i\rangle. \tag{4}$$

Here $\mu = \pm 1, 0$ denotes spherical components of $\boldsymbol{\Psi}$ and $\boldsymbol{\alpha}$. In the following discussion we restrict ourselves to the $ns_{1/2}$ states,

$$|i\rangle = \begin{bmatrix} f_n(r)\Omega_{1/2,0,m}(\boldsymbol{n}) \\ i\alpha g_n(r)\Omega_{1/2,1,m}(\boldsymbol{n}) \end{bmatrix}, \quad |f\rangle = \begin{bmatrix} f_{n'}(r)\Omega_{1/2,0,m'}(\boldsymbol{n}) \\ i\alpha g_{n'}(r)\Omega_{1/2,1,m'}(\boldsymbol{n}) \end{bmatrix},$$

where $\alpha = 1/c \approx 1/137$ is the fine structure constant and $f_n(r)$, $g_n(r)$, and $\Omega_{1/2,l,m}(\boldsymbol{n})$ are, respectively, the radial and spin-angular parts of the initial $|ns_{1/2}\rangle$ state with the principal quantum number $n = 1, 2, \ldots$; appropriate primed quantities will hereafter refer to the final $n's_{1/2}$ state. To separate angular variables in Eq. (2) we write

$$\Psi_\mu(E) = \Psi_\mu^{(1/2)}(E) + \Psi_\mu^{(3/2)}(E) \tag{5}$$

and assume that

$$\Psi_\mu^{(1/2)}(E) = (-1)^{1/2+\mu+m}\sqrt{\frac{2}{3}}\begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ -\mu - m & \mu & m \end{pmatrix} \times \frac{1}{r}\begin{bmatrix} i\xi_1(r;\varepsilon)\Omega_{1/2,1,-\mu-m}(\boldsymbol{n}) \\ \alpha\eta_1(r;\varepsilon)\Omega_{1/2,0,-\mu-m}(\boldsymbol{n}) \end{bmatrix}, \tag{6a}$$

$$\Psi_\mu^{(3/2)}(E) = (-1)^{1/2+\mu+m}\frac{4}{\sqrt{3}}\begin{pmatrix} 3/2 & 1 & 1/2 \\ -\mu - m & \mu & m \end{pmatrix} \times \frac{1}{r}\begin{bmatrix} i\xi_2(r;\varepsilon)\Omega_{3/2,1,-\mu-m}(\boldsymbol{n}) \\ \alpha\eta_1(r;\varepsilon)\Omega_{3/2,2,-\mu-m}(\boldsymbol{n}) \end{bmatrix}, \tag{6b}$$

$\begin{pmatrix}\cdots\\ \cdots\end{pmatrix}$ being the usual $3j$ symbols and $\varepsilon = E - c^2$. In terms of $\varrho = 2\lambda r$, $\lambda = \sqrt{-\varepsilon(2 + \alpha^2\varepsilon)}$, $\gamma_1 = \sqrt{1 - (\alpha Z)^2}$, and $\gamma_2 = \sqrt{4 - (\alpha Z)^2}$, a system of first-order differential equations satisfied by $\xi_j(r;\varepsilon)$, $\eta_j(r;\varepsilon)$, $j = 1, 2$ can be uncoupled by means of the substitutions [2],

$$\begin{bmatrix} \xi_j(r;\varepsilon) \\ \eta_j(r;\varepsilon) \end{bmatrix} = \varrho^{\gamma_j}e^{-\varrho/2}\begin{bmatrix} P_j(\varrho) + Q_j(\varrho) \\ -\frac{\varepsilon}{\lambda}[P_j(\varrho) - Q_j(\varrho)] \end{bmatrix}, \tag{7}$$

to result eventually in the following hypergeometric-type equations for the four auxiliary functions, $P_j(\varrho)$, $Q_j(\varrho)$:

$$\begin{bmatrix} P_j''(\varrho) + (a_j - \varrho)P_j'(\varrho) - b_jP_j(\varrho) \\ Q_j''(\varrho) + (a_j - \varrho)Q_j'(\varrho) - (b_j + 1)Q_j(\varrho) \end{bmatrix} = \varrho^{\gamma_1-\gamma_j}e^{\beta\varrho}\sum_{m=0}^{n}\varrho^m\begin{bmatrix} C_m^{(j)} \\ B_m^{(j)} \end{bmatrix}. \tag{8}$$

Here $a_j = 2\gamma_j + 1$, $b_j = \gamma_j - (Z/\lambda)(1 + \alpha^2\varepsilon)$, $\beta = (\lambda N - Z)/(2\lambda N)$, and $N = \sqrt{n^2 - 2(n-1)(1-\gamma_1)}$. For any $n = 1, 2, \ldots$ and $j = 1, 2$ each function of the pair, $\{P_j(\varrho), Q_j(\varrho)\}$, can be expressed in terms of the other one. So are two ρ-independent sets of coefficients, $\{C_m^{(j)}\}_{m=0}^n$ and $\{B_m^{(j)}\}_{m=0}^n$, whose general forms are given by elementary yet bulky expressions [2]. Together with obvious symmetry properties of Eqs. (8), this enables all four functions to be determined simultaneously by using proper analytical solution to any single equation. To fix it uniquely, we impose the following boundary conditions:

$$\begin{bmatrix} \xi_j(r;\varepsilon) \\ \eta_j(r;\varepsilon) \end{bmatrix} = \begin{cases} 0, & \text{if } r \to 0, \\ o\big[e^{(\lambda - \frac{Z}{N})r}\big], & \text{if } r \to \infty, \end{cases} \tag{9}$$

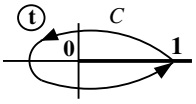
that follow directly from a requirement that the LRRWF, $\Psi(E)$, of Eqs. (3)–(6) should remain finite for any E. After some elaborate algebra whose details will be given elsewhere, four solutions can be compactly expressed as

$$\begin{bmatrix} P_j(\varrho) \\ Q_j(\varrho) \end{bmatrix} = \sum_{m=0}^{n}\begin{bmatrix} C_m^{(j)}S_m^{(j)}(\varrho, z; b_j) \\ B_m^{(j)}S_m^{(j)}(\varrho, z; b_j + 1) \end{bmatrix}, \tag{10}$$

where $S_m^{(j)}(\varrho, z; \delta)$ satisfies the recurrence relation, $S_{m+1}^{(j)}(\varrho, z; \delta) = (1 + z)^2(\partial/\partial z)S_m^{(j)}(\varrho, z; \delta)$, and has the form

$$S_m^{(j)}(\varrho, z; \delta) = -\frac{\Gamma(\gamma_j^+ + m + 1)}{\Gamma(2\gamma_j + 1)}(1 + z)^{\gamma_j^+ + 1 + m} \times \int_0^1 \exp\!\left(\frac{\varrho z t}{1 + tz}\right)t^{\delta-1}(1 - t)^{m-\gamma_j^-}(1 + zt)^{-\gamma_j^+ - m - 1} \times {}_1F_1\!\left[\gamma_j^- - m, 2\gamma_j + 1; -\frac{\varrho(1 + z)t}{(1 + zt)(1 - t)}\right]dt. \tag{11}$$

Here $\gamma_j^\pm = \gamma_j \pm \gamma_1$ so that $\gamma_1^- = 0$, $z = (\lambda N - Z)/(\lambda N + Z)$ and ${}_1F_1(\ldots)$ is the confluent hypergeometric function [18]. For n and ε such that $b_j < -1$, the integral in Eq. (11) formally diverges.



Its analytical continuation to all n and the entire complex ε plane can be carried through by replacing the definite integral with a contour one, $\int_0^1 \ldots dt \to [\exp(2\pi i\delta) - 1]^{-1}\oint_C \ldots dt$, taken along the path C shown above. Equations (10) and (11) generalize appropriate nonrelativisitic results [3,19] that can be retrieved by setting $\gamma_1 = 1$, $\gamma_2 = 2$, $N = n$, $\varepsilon = -Z^2/(2\nu^2n^2)$, $\lambda = Z/(\nu n)$, $z = (1 - \nu)/(1 + \nu)$, $b_2 = 2 - \nu n$, and $b_1 = 1 - \nu n$. The quantity $S_m(\varrho, z; \delta)$ in Eq. (10) can also be expressed further in terms of the Appel Φ_1 functions [20]. The integral representation of Eq. (11) appears, however, to be much more convenient for most practical applications,

both analytical and numerical. This last feature is illustrated in Fig. 1 where, as a typical example, we present graphs of all four radial components of the hydrogenic ($Z = 1$) $2s_{1/2}$ state LRRWF calculated for $r = 0, \ldots, 15$ and $0 \le \omega = \varepsilon - \varepsilon_{2s_{1/2}} \le \varepsilon_{3s_{1/2}} - \varepsilon_{2s_{1/2}}$.

As an important byproduct application of the LRRWF, Eqs. (3)–(11) are now used to calculate analytically the RDTLS for the hydrogenic $|i\rangle = |ns_{1/2}\rangle$ and $|f\rangle = |n's_{1/2}\rangle$ states, in which case (also assuming $m' = m$) the tensor $[C_{kl}(\omega, \omega')]_{n's,ns}$ can be expressed as

$$[C_{kl}(\omega, \omega')]_{n'n} = -\frac{\delta_{kl}}{9\omega\omega'}[T_{n'n}(\omega) + T_{n'n}(-\omega')], \quad (12)$$

with δ_{kl} being the Kroneckers's symbol. It is convenient to split the scalar function $T_{n'n}(\omega)$ into two parts,

$$T_{n'n}(\omega) = T^{(1/2)}_{n'n}(\omega) + 8\,T^{(3/2)}_{n'n}(\omega), \quad (13)$$

and express [2] $T^{[(1/2),(3/2)]}_{n'n}$ in terms of $\xi_j(r;\varepsilon)$, $\eta_j(r;\varepsilon)$ and $f_{n'}(r)$, $g_{n'}(r)$. Subsequent cumbersome but quite straightforward calculation finally yields

$$T^{(1/2)}_{n'n} = -\frac{2\varepsilon}{\lambda + Z}\left[\sum_{m'=0}^{n'}\sum_{m=0}^{n} B^{(1)}_{m'}B^{(1)}_{m}S^{(1)}_{m'm}(z', z) + F^{(1/2)}_{n'n}\right],$$

$$T^{(3/2)}_{n'n} = -\frac{2\varepsilon}{2\lambda - Z}\left[\sum_{m'=0}^{n'}\sum_{m=0}^{n} B^{(2)}_{m'}B^{(2)}_{m}S^{(2)}_{m'm}(z', z) + F^{(3/2)}_{n'n}\right].$$

Here $B^{(j)}_{m'}$, $B^{(j)}_{m}$ have already been introduced above and

$$F^{(1/2)}_{n'n} = \frac{1}{4\varepsilon^2}\int_0^\infty r^3[3\varepsilon f_{n'}(r) + \lambda g_{n'}(r)] \times [3\varepsilon f_n(r) + \lambda g_n(r)]dr, \quad (14a)$$

$$F^{(3/2)}_{n'n} = \frac{\lambda^2}{4\varepsilon^2}\int_0^\infty r^3 g_{n'}(r)g_n(r)dr. \quad (14b)$$

The function $S^{(j)}_{m'm}(z', z) = S^{(j)}_{mm'}(z, z')$ satisfies recurrence relations: $S^{(j)}_{m'+1,m}(z', z) = (1 + z')^2(\partial/\partial z')S^{(j)}_{m'm}(z', z)$, $S^{(j)}_{m',m+1}(z', z) = (1 + z)^2(\partial/\partial z)S^{(j)}_{m'm}(z', z)$, and it is given by

$$S^{(j)}_{m'm}(z', z) = -\frac{\Gamma(\gamma_j^+ + m' + 1)\Gamma(\gamma_j^+ + m + 1)}{\Gamma(2\gamma_j + 1)} \times (1 + z')^{\gamma_j^+ + 1 + m'}(1 + z)^{\gamma_j^+ + 1 + m}[\exp(2\pi i b_j) - 1]^{-1} \times \oint_C dt\, t^{b_j}\frac{(1 + z't)^{m - \gamma_j^-}(1 + zt)^{m' - \gamma_j^-}}{(1 - z'zt)^{2\gamma_1 + m' + m + 1}} \times {}_2F_1\left[\gamma_j^- - m', \gamma_j^- - m; 2\gamma_j + 1; \frac{t(1 + z')(1 + z)}{(1 + z't)(1 + zt)}\right], \quad (15)$$

where $\varepsilon = \varepsilon_{ns_{1/2}} + \omega$ is assumed in all formulas above, $z' = (\lambda N' - Z)/(\lambda N' + Z)$ and ${}_2F_1(\ldots)$ is the hypergeometric function [18]. A form of $S^{(1)}_{m'm}(z', z)$ for any m', m can be found by acting with $(1 + z)^2(\partial/\partial z)$ and $(1 + z')^2 \times (\partial/\partial z')$ on $S^{(1)}_{00}(z', z) = -[\Gamma(2\gamma_1 + 1)/b_1][(1 + z') \times (1 + z)]^{2\gamma_1 + 1}{}_2F_1(2\gamma_1 + 1, b_1; b_1 + 1; z'z)$. The same method is also applicable to $j = 2$. Although Eq. (15) is already well suited in this case as it is, it may be used to derive a number of as yet unknown series representations for $S^{(2)}_{m'm}(z', z)$. These can be obtained by replacing the ${}_2F_1$ function with the Mellin-Barnes integral [18] to yield expansions converging — unlike that of Refs. [8,9] — in $|z'| \le 1$, $|z| \le 1$, i.e., enabling $C_{kl}(\omega, \omega')$ to be calculated at $\omega', \omega > I_{ns}$. Of particular interest in this context are diagonal $ns_{1/2} \to ns_{1/2}$ transitions ($z' = z$), in which case RDTLS reduces to the relativistic dipole dynamic

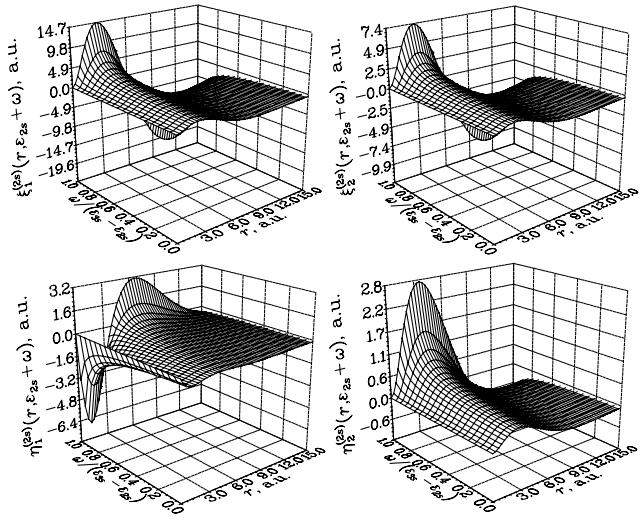


FIG. 1. Four radial components, $\xi_{1,2}(r; \varepsilon_{2s_{1/2}} + \omega)$ and $\eta_{1,2}(r; \varepsilon_{2s_{1/2}} + \omega)$, of the LRRWF for the $2s_{1/2}$ state of a hydrogenic ion with $Z = 1$ versus radial distance, r, and photon energy $\omega \le \varepsilon_{3s_{1/2}} - \varepsilon_{2s_{1/2}}$.

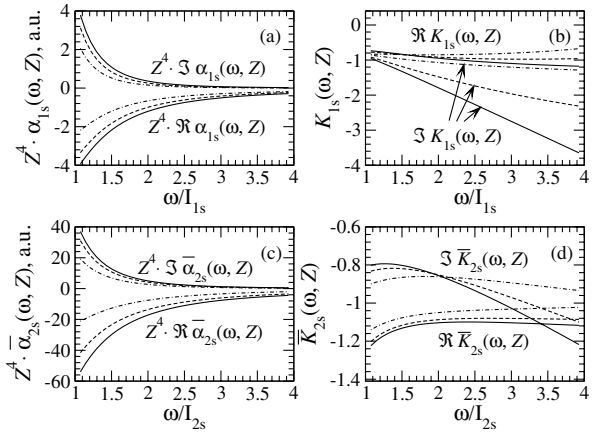


FIG. 2. (a),(c): RDDPs, $Z^4\alpha_{1s}(\omega, Z)$ and $Z^4\overline{\alpha}_{2s}(\omega, Z)$, of the hydrogenic $1s_{1/2}$ and $2s_{1/2}$ states with $Z = 1$ (solid line), $Z = 60$ (dashed line) and $Z = 100$ (dash-dotted line) as functions of $\omega > I_{1s}, I_{2s}$; (b),(d): ω and Z dependence of the functions $K_{ns}(\omega, Z)$: $\alpha_{ns}(\omega, Z) = \alpha^{(\mathrm{nr})}_{ns}(\omega, Z)[1 + K_{ns}(\omega, Z)\,(\alpha Z)^2]$, $n = 1, 2$.

polarizability (RDDP) of the $ns_{1/2}$ level, $\alpha_{ns}(\omega) \equiv [C_{kl}(\omega, \omega)]_{ns,ns}/\delta_{kl}$. The utility of our analytic technique is demonstrated in Fig. 2 where we display results of RDDP calculations for the $1s_{1/2}$ and $2s_{1/2}$ states at ω lying above respective ionization thresholds [for the $2s_{1/2}$ level RDDP, see Eq. (18)].

Finally, we present here new analytic results for the *static* RDP of the lowest $ns_{1/2}$ states, as being of fundamental interest in their own right. The calculations have been carried out by first setting $\omega' = \omega$ in Eqs. (12) and (13) and then taking the limit $\omega \to 0$. After some elaborate technical work, we get eventually:

$$Z^4\alpha_{1s}(0) = \frac{(2-\gamma_1)^2\Gamma^2(\gamma_2+\gamma_1+2)}{18(\gamma_1-\gamma_2-1)\Gamma(2\gamma_2+1)\Gamma(2\gamma_1+1)}\,{}_3F_2(\gamma_2^- + 1, \gamma_2^- - 1, \gamma_2^- - 1; \gamma_2^- + 2, 2\gamma_2 + 1; 1)$$
$$+ \frac{1}{36}(2\gamma_1+1)(1+\gamma_1)(12+13\gamma_1+4\gamma_1^2) \tag{16}$$

$$= \frac{9}{2}\cdot\left[1 - \frac{28}{27}(\alpha Z)^2 + \frac{31+2\pi^2}{432}(\alpha Z)^4 - \frac{6607 - 672\pi^2 - 5184\zeta(3) + 41472\delta_1}{746496}(\alpha Z)^6 + \ldots\right], \tag{17}$$

$$Z^4\overline{\alpha}_{2s}(0) = -\frac{2Z^4}{\alpha^2\Delta\varepsilon^3}\sum_{m'=-3/2}^{3/2}|\langle 2s_{1/2}|\alpha_z|2p_{3/2}\rangle|^2 - \frac{N^5(3N-4)^2[2\gamma_2(N^4-2N^2+4)+N^6-4N^4+14N^2-8]}{384(2-N)(N-1)\Gamma^{-2}(\gamma_2+\gamma_1+1)\Gamma(2\gamma_2+1)\Gamma(2\gamma_1+1)} \tag{18}$$
$$\times {}_3F_2(\gamma_2^-, \gamma_2^- - 1, \gamma_2^- - 1; \gamma_2^- + 1, 2\gamma_2 + 1; 1) - \frac{N^5(N+1)(7N^6-28N^5+53N^4-58N^3+20N^2-136N+16)}{192(2-N)}$$

$$= 120\cdot\left[1 - \frac{367}{240}(\alpha Z)^2 - \frac{1992\pi^2 - 51840\zeta(3) + 10368\delta_2 - 36125}{138240}(\alpha Z)^4 + \ldots\right]. \tag{19}$$

Here $N = \sqrt{2(\gamma_1+1)}$, $\zeta(3) = 1.202\,06\ldots$ is the value of the Riemann zeta function [18], $\Delta\varepsilon = \varepsilon_{2p_{3/2}} - \varepsilon_{2p_{1/2}}$, and ${}_3F_2(\ldots;1)$ denotes the hypergeometric function of the argument 1; $\delta_1 = 0.01620\ldots$ and $\delta_2 = -0.08016\ldots$ stand for numerical sums of two elementary series. The forms of Eqs. (16) and (17) are in accord with that formerly obtained in Ref. [6] by making use of the relativisitic Green's function in the r gauge. Also, their values are in agreement with those computed in Ref. [10] within the framework of a basis set method. The first extra term in Eq. (18) has been added so as to remove a spurious part of the total $2p_{3/2}$ contribution to the $2s_{1/2}$ state RDDP, $2/(\alpha^2\Delta\varepsilon)[1/(\Delta\varepsilon^2-\omega^2)+1/\omega^2]\sum_{m'=-3/2}^{3/2}\times|\langle 2s_{1/2}|\alpha_z|2p_{3/2}\rangle|^2$, which is singular [$\simeq 384\alpha^{-2}/Z^6$] as $\alpha \to 0$ at $\omega = 0$. Thus regularized RDDP, $\overline{\alpha}_{2s}(0)$, is meaningful at $\alpha = 0$, thereby enabling its nonrelativisitic limit [11,17] to be retrieved [cf. Eq. (19)]. Among other potential applications of the LRRWF approach is a wide spectrum of low-energy scattering and laser-induced phenomena occurring with few-body atoms and molecules. These studies are in progress and their results will be reported in due course.

The author is indebted to Martin Jungen for stimulating discussions on the problem. Financial support provided by the Schweizerischer Nationalfonds (Projects No. 21-65011.01 and No. 20-65290.01) is acknowledged.

*Electronic address: Victor.Yakhontov@unibas.ch

[1] N. B. Delone and V. P. Krainov, *Multiphoton Processes in Atoms* (Springer-Verlag, Berlin, 1994).
[2] V. Yakhontov, Can. J. Phys. **80**, 1413 (2002).
[3] V. Yakhontov, R. Santra, and K. Jungmann, J. Phys. B **32**, 1615 (1999).
[4] V. Meyer *et al.*, Phys. Rev. Lett. **84**, 1136 (2000).
[5] V. G. Ivanov and S. Karshenboim, Sov. Phys. JETP **112**, 805 (1997).
[6] B. A. Zon, N. L. Manakov, and L. P. Rapoport, Sov. J. Nucl. Phys. **15**, 282 (1972).
[7] G. W. F. Drake and S. P. Goldman, Adv. At. Mol. Phys. **25**, 393 (1988).
[8] V. Florescu, M. Marinescu, and R. H. Pratt, Phys. Rev. A **42**, 3844 (1990).
[9] K. Pachucki, Phys. Rev. A **46**, 648 (1992).
[10] R. Szmytkowski, J. Phys. B **30**, 825 (1997).
[11] A. A. Krylovetsky, N. L. Manakov, and S. I. Marmo, Laser Phys. **7**, 781 (1997).
[12] L. A. Thu *et al.*, J. Phys. B **29**, 2897 (1996).
[13] F. A. Parpia and W. R. Johnson, Phys. Rev. A **26**, 1142 (1982).
[14] B. Podolsky, Proc. Natl. Acad. Sci. U.S.A. **14**, 253 (1928).
[15] R. M. Sternheimer, Phys. Rev. **96**, 951 (1954).
[16] A. Dalgarno and J. T. Lewis, Proc. R. Soc. London A **233**, 70 (1955).
[17] V. Yakhontov and K. Jungmann, Z. Phys. D **38**, 141 (1996).
[18] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1970).
[19] V. Florescu and T. Marian, Phys. Rev. A **34**, 4641 (1986).
[20] E. H. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher Transcendental Functions* (McGraw-Hill, New York, 1953).