Positivity Constraints on Initial Spin Observables in Inclusive Reactions

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For any inclusive reaction of the type $A_1(\text{spin } 1/2) + A_2(\text{spin } 1/2) \to B + X$, we derive new positivity constraints on spin observables and study their implications for theoretical models in view, in particular, of accounting for future data from the polarized pp collider at Brookhaven RHIC. We find that the single transverse spin asymmetry A_N, in the central region for several processes, for example, jet production, direct photon production, and lepton-pair production, is expected to be such that $|A_N| \lesssim 1/2$, rather than the usual bound $|A_N| \leq 1$.

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Let us consider an inclusive reaction of the type

$$A_1(\text{spin } 1/2) + A_2(\text{spin } 1/2) \to B + X, \qquad (1)$$

where the spins of both initial spin 1/2 particles can be in any possible directions and no polarization is observed in the final state. The observables of this reaction, which are the spin-dependent differential cross sections with respect to the momentum of B, can be expressed in terms of the discontinuities (with respect to the invariant mass squared of X) of the amplitudes for the forward three-body scattering

$$A_1 + A_2 + \overline{B} \to A_1 + A_2 + \overline{B}, \qquad (2)$$

as given by the generalized optical theorem. We assume parity conservation, so the complete knowledge of this reaction requires the determination of *eight* real functions, which is the number of independent spin observables [1]. In order to define these observables, we recall the standard notation used in Ref. [2] $(A_1A_2 \mid BX)$, by which the spin directions of A_1, A_2, B, and X are specified in one of the three possible directions L, N, S. Since the final spins are not observed, we have in fact $(A_1A_2 \mid 00)$ and $\mathbf{L}$, $\mathbf{N}$, $\mathbf{S}$ are unit vectors, in the center-of-mass system, along the incident momentum, along the normal to the scattering plane, which contains A_1, A_2, and B, and along $\mathbf{N} \times \mathbf{L}$, respectively. In addition to the unpolarized cross section $\sigma_0 = (00 \mid 00)$, there are *seven* spin-dependent observables, *two* single transverse spin asymmetries

$$A_{1N} = (N0 \mid 00) \quad \text{and} \quad A_{2N} = (0N \mid 00), \qquad (3)$$

and *five* double-spin asymmetries

$$A_{LL} = (LL \mid 00), \quad A_{SS} = (SS \mid 00), \quad A_{NN} = (NN \mid 00),$$
$$A_{LS} = (LS \mid 00), \quad \text{and} \quad A_{SL} = (SL \mid 00). \qquad (4)$$

The state of polarization of the two spin 1/2 particles A_1 and A_2 is characterized by the 2×2 density matrices ρ_1 and ρ_2 defined as

$$\rho_i = \tfrac{1}{2}(\mathbf{I}_2 + \mathbf{e}_i \cdot \boldsymbol{\sigma}), \qquad i = 1, 2, \qquad (5)$$

where $\mathbf{e}_1$ and $\mathbf{e}_2$ are the polarization unit vectors of A_1 and A_2, $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ stands for the three 2×2 Pauli matrices, and $\mathbf{I}_2$ is the 2×2 unit matrix. The state of polarization of the incoming system in the reaction (1) is described by the 4×4 density matrix ρ, which is the direct product $\rho = \rho_1 \otimes \rho_2$.

The spin-dependent cross section corresponding to (1) is

$$\sigma(\mathbf{e}_1, \mathbf{e}_2) = \mathrm{Tr}(M\rho), \qquad (6)$$

where M denotes the 4×4 cross section matrix which we parametrize in the following way [3]:

$$M = \sigma_0[\mathbf{I}_4 + A_{1N}\sigma_{1z} \otimes \mathbf{I}_2 + A_{2N}\mathbf{I}_2 \otimes \sigma_{2z} + A_{NN}\sigma_{1z} \otimes \sigma_{2z} + A_{LL}\sigma_{1x} \otimes \sigma_{2x} + A_{SS}\sigma_{1y} \otimes \sigma_{2y} + A_{LS}\sigma_{1x} \otimes \sigma_{2y} + A_{SL}\sigma_{1y} \otimes \sigma_{2x}]. \qquad (7)$$

Here $\mathbf{I}_4$ is the 4×4 unit matrix and σ_0 stands for the spin-averaged cross section. This expression is fully justified, since we have explicitly

$$\sigma(\mathbf{e}_1, \mathbf{e}_2) = \sigma_0[1 + A_{1N}e_{1z} + A_{2N}e_{2z} + A_{NN}e_{1z}e_{2z} + A_{LL}e_{1x}e_{2x} + A_{SS}e_{1y}e_{2y} + A_{LS}e_{1x}e_{2y} + A_{SL}e_{1y}e_{2x}]. \qquad (8)$$

The crucial point is that M is a Hermitian and *positive* matrix and in order to derive the positivity conditions one should write the explicit expression of M as given by Eq. (7). Then one observes that by permuting two rows and two columns, it reduces to the simple form

$$\left(\begin{array}{c|c} M_+ & 0 \\ \hline 0 & M_- \end{array}\right),$$

where $M_\pm$ are 2×2 Hermitian matrices which must be positive, leading to the following *two* strongest constraints [5]:

$$(1 \pm A_{NN})^2 \geq (A_{1N} \pm A_{2N})^2 + (A_{LL} \pm A_{SS})^2 + (A_{LS} \pm A_{SL})^2. \quad (9)$$

These are necessary and sufficient conditions and we note that if, for a given reaction, one has $A_{NN} = \mp 1$, Eq. (9) implies $A_{1N} = \pm A_{2N}$, $A_{LL} = \pm A_{SS}$, $A_{LS} = \pm A_{SL}$, and $1 \geq A_{1N}^2 + A_{LL}^2 + A_{LS}^2$. As special cases of Eq. (9), we have the *six* weaker constraints

$$1 \pm A_{NN} \geq |A_{1N} \pm A_{2N}|, \quad (10)$$

$$1 \pm A_{NN} \geq |A_{LL} \pm A_{SS}|, \quad (11)$$

and

$$1 \pm A_{NN} \geq |A_{LS} \pm A_{SL}|. \quad (12)$$

These constraints are very general [9] and must hold in any kinematical region and for many different situations such as electron-proton scattering, electron-positron scattering, or quark-quark scattering, but we now turn to a specific case, which is of direct relevance to the spin program at the Brookhaven RHIC polarized pp collider [10]. Now let us consider a proton-proton collision and let us call y the rapidity of the outgoing particle B. In this case since the initial particles are identical, we have $A_{1N}(y) = A_{2N}(-y)$ and $A_{LS}(y) = A_{SL}(-y)$ [11]. In this case Eq. (9), which becomes two constraints among five independent spin observables, reads

$$[1 \pm A_{NN}(y)]^2 \geq [A_{1N}(y) \pm A_{1N}(-y)]^2 + [A_{LL}(y) \pm A_{SS}(y)]^2 + [A_{LS}(y) \pm A_{LS}(-y)]^2. \quad (13)$$

This implies, in particular, for $y = 0$,

$$1 + A_{NN}(0) \geq 2|A_N(0)|, \quad (14)$$

and

$$1 + A_{NN}(0) \geq 2|A_{SL}(0)|, \quad (15)$$

so that the allowed range of A_N and A_{SL} is strongly reduced, if A_{NN} turns out to be large and negative. Conversely, if $A_{NN} \simeq 1$, these constraints are useless. Note that, in the kinematical region accessible to the pp polarized collider, a calculation of A_{NN} for direct photon production and jet production has been performed [12]; it was found that $|A_{NN}|$ is of the order of 1% or 2%. Similarly, based on Ref. [13], this double transverse spin asymmetry for lepton-pair production was estimated to be a few percent [14]. The direct consequence of these estimates is that $|A_N|$ and $|A_{SL}|$, for these processes [15], are essentially bounded by 1/2. In addition, from Eq. (11), there are two other nontrivial constraints: $1 \geq |A_{LL} \pm A_{SS}|$.

Single transverse spin asymmetries in inclusive reactions at high energies are now considered to be directly related to the transverse momentum of the fundamental partons involved in the process. This new viewpoint, which has been advocated to explain the existing data in semi-inclusive deep inelastic scattering [17,18], will have to be more firmly established also by means of future data from Brookhaven RHIC. On the theoretical side, several possible leading-twist QCD mechanisms [19,20] have been proposed to generate these asymmetries in leptoproduction [21,22], but also in pp collisions. We believe that these new positivity constraints on spin observables for a wide class of reactions will be of interest for model builders as well as for future measurements.

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