Scalings of Domain Wall Energies in Two Dimensional Ising Spin Glasses

C. Amoruso,[1] E. Marinari,[1] O. C. Martin,[2] and A. Pagnani[2]
[1]*Dipartimento di Fisica, SMC and UdR1 of INFM, INFN, Università di Roma "La Sapienza," Piazzale Aldo Moro 2, 00185 Roma, Italy*
[2]*Laboratoire de Physique Théorique et Modèles Statistiques, bâtiment 100, Université Paris-Sud, F-91405 Orsay, France*
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We study domain wall energies of two dimensional spin glasses. The scaling of these energies depends on the model's distribution of quenched random couplings, falling into *three* different classes. The first class is associated with the exponent $\theta \approx -0.28$; the other two classes have $\theta = 0$, as can be justified theoretically. In contrast to previous claims, we find that $\theta = 0$ does not indicate $d = d_l^c$ but rather $d \leq d_l^c$, where d_l^c is the lower critical dimension.

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Spin glasses [1] exhibit many subtle phenomena such as diverging nonlinear susceptibilities, aging, and memory, making it a real challenge to understand these materials. In spite of much work, there is still no consensus even on the nature of the frozen order in equilibrium. More surprising still, the case of two dimensions also is not completely understood. In particular, the scaling of the stiffness, a cornerstone of spin glass theory, is different when the spin-spin couplings are of the form $J_{ij} = \pm 1$ compared to when they have a Gaussian distribution [2]. This has been confirmed since using more powerful numerical techniques [3,4], and in fact it was interpreted in [4] as a lack of universality, but this is unexpected and unexplained. Here we solve this puzzle: we find that different types of quenched disorder lead to three distinct behaviors. In particular, we motivate why the class of models that includes the case $J_{ij} = \pm 1$ gives for the stiffness exponent $\theta = 0$, and we explain what θ tells us about the lower critical dimension.

The model, its properties, and our methods.—The model consists of $N = L^2$ Ising spins $S_i = \pm 1$ on a simple square lattice with periodic boundary conditions. The Hamiltonian is

$$H \equiv -\sum_{\langle ij \rangle} S_i J_{ij} S_j, \tag{1}$$

where the sum runs over all pairs of nearest neighbors $\langle ij \rangle$ and the J_{ij} are the quenched random spin-spin couplings. We consider different distributions of these couplings, all of which are symmetrical about $J = 0$. We begin with continuous distributions; most common is the one where the J_{ij} are Gaussian random variables with zero mean and unit variance. After that we investigate discrete distributions; the most common distribution of this type has $J_{ij} = \pm 1$ with equal probability.

An important feature of spin glass ordering is the spin glass stiffness; the corresponding exponent θ describes how excitation free energies scale with the associated length scale. The standard way to measure this exponent is via the change in the system's free energy when going from periodic to antiperiodic boundary conditions. At $T = 0$ this reduces to measuring the difference

$$\delta E = E_0^{(P)} - E_0^{(AP)}, \tag{2}$$

where $E_0^{(P)}$ and $E_0^{(AP)}$ are the ground state energies for the system with, respectively, periodic and antiperiodic boundary conditions, say, in the x direction. We are interested in the probability distribution of δE when considering an ensemble of J_{ij} and in the scaling law of its standard deviation ΔE:

$$\Delta E \underset{L \to \infty}{\sim} L^{\theta}. \tag{3}$$

Measurements of θ in two dimensional spin glasses (see, for instance, [2]) give $\theta \approx -0.28$. However, for the $J_{ij} = \pm 1$ distribution, Hartmann and Young [4] recently showed that ΔE remains of $O(1)$ for increasing L, implying that in this case $\theta \approx 0$. In dimension d above the lower critical dimension d_l^c we have $\theta > 0$ and spin glass ordering is stable against thermal fluctuations. On the contrary, when $\theta < 0$, thermal fluctuations prevent spin glass ordering. Because of this, the authors of [4] conjectured that $d_l^c = 2$ for the $J_{ij} = \pm 1$ model. We shall see that d_l^c should be identified with the highest value of d where $\theta \leq 0$, and so in fact $d_l^c \approx 2.5$ as believed before the study in [4].

In this work we address these questions by first determining numerically the properties of $P(\delta E)$ and then by using the real space renormalization group picture. For the first part, we compute the ground states of our systems using a heuristic algorithm [5]. In practice, when the lattice is not too large ($L \leq 80$), the algorithm returns the ground state with a high level of confidence for all of the distributions we consider in this work. The problem is to reduce the statistical errors enough; in practice we used a few tens of thousands of samples at a few values of L for each case.

Class 1: "continuous" distributions.—We first focus on distributions $P(J)$ that include a continuous part (we see later that this class includes certain discrete distributions

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also). When L is sufficiently large, δE can then take on arbitrary values. The value of θ for continuous distributions is well known only for Gaussian J_{ij}; in fact, we are aware of no tests of universality in $d = 2$, though the standard lore is that both θ and the shape of $P(\delta E)$ are universal [2].

In a first series of runs we obtained $P(\delta E)$ and ΔE for the model with Gaussian couplings. Then we moved on to a continuous yet singular probability density $P(J_{ij})$: $P(J_{ij} = J) = fP_1(J) + (1 - f)P_2(J)$, where $P_1(J) \equiv [e^{(J-1)^2/2} + e^{(J+1)^2/2}]/\sqrt{8\pi}$, $P_2(J) \equiv [\delta(J-1) + \delta(J+1)]/2$, and f is a measure of the height of the distribution at $J \approx 0$. We refer to this $P(J_{ij})$ as the *broadened bimodal (BB) distribution* since it reduces to the $J_{ij} = \pm 1$ distribution when $f = 0$.

In Fig. 1 we show ΔE as a function of L when $P(J_{ij})$ is (i) a Gaussian of zero mean and unit variance (*GAUSS* data); (ii) the BB distribution, with $f = 0.1$ (*BB* 0.1 data); (iii) as in (ii) but with $f = 0.2$ (*BB* 0.2 data); (iv) Gaussian but with the part in the interval $[-0.5, 0.5]$ forced to be 0 (*HOLE* data). Note that this last distribution has a large gap around $J_{ij} = 0$. In the Gaussian case the power law scaling of ΔE can be determined with good accuracy already from quite small lattices; fits to these data lead to $\theta = -0.282 \pm 0.004$, in agreement with previous work. The distributions (ii), (iii), and (iv) give rise to a similar scaling, albeit only at larger L values. We have also considered other distributions such as $P(J_{ij})$ uniform in $[-1.5, -0.5] \cup [0.5, 1.5]$ (notice that this distribution also has a gap around $J = 0$), obtaining similar results. It thus seems very reasonable to expect that *all* distributions with a continuous part will lead to the same exponent, $\theta \approx -0.28$.

A second universality issue concerns the *shape* of $P(\delta E)$. In the inset in Fig. 1 we show the probability density $P(|\delta E|/\Delta E)$ when $L = 60$ for the *BB* 0.1, *BB* 0.2, and *GAUSS* data: the different data sets basically coincide within statistical errors, strengthening the claim that in this class the distribution of domain wall energies is universal (the curve displayed is just to guide the eye).

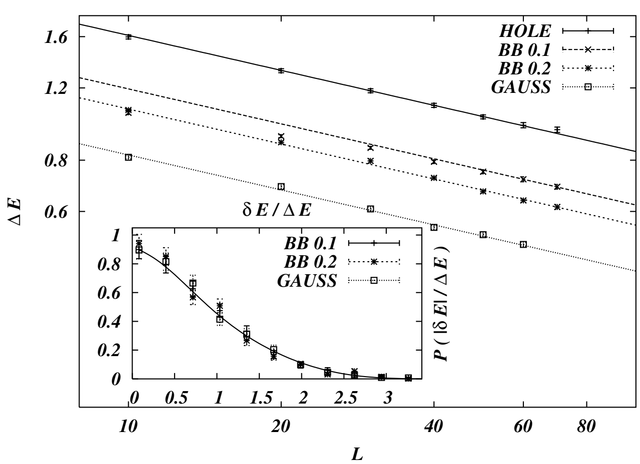

FIG. 1. ΔE as a function of the system size for four different J_{ij} distributions. Straight lines are best one-parameter fits of the form const $\times L^{-0.282}$. Inset: the probability distribution $P(|\delta E|/\Delta E)$ at $L = 40$ for three of these distributions.

Class 2: quantized energies.—At variance with the former distributions, the $J_{ij} = \pm 1$ model leads to $\theta \sim 0$ [4]. We show in Fig. 2 that in this model ΔE saturates quickly as L grows. Is the $J_{ij} = \pm 1$ model a special case, a class on its own? The crucial point is that the possible δE values are *quantized*: δE is always a multiple of a *quantum* Q; here $Q = 4$. This led us to consider distributions other than the ± 1 one with this same quantization property. We begin by "diluting" the $J_{ij} = \pm 1$ model, setting $J_{ij} = 0$ with probability 0.2. The main effect of this is to reduce the quantum from 4 to 2; indeed, the local fields now can take the value $0, 1, 2, 3, 4$ instead of $0, 2, 4$. In Fig. 2 we see that for this model [diluted (*DIL*) data] ΔE seems to saturate, so again $\theta = 0$. However, the convergence is slow. In any renormalization group picture this convergence is governed by a "correction to scaling" exponent ω. We assume $\theta = 0$ and that the asymptotic value of ΔE is a nonzero constant given by the $J = \pm 1$ data; then we fit the DIL model to the form:

$$\Delta E(L) \approx \Delta E(L = \infty) + AL^{-\omega}. \tag{4}$$

with A and ω adjustable parameters. We have also considered distributions where $J_{ij} = \pm J_1$ or $\pm J_2$ with equal probability (we have studied the cases $J_2/J_1 = 1.5$, 2, and 3). Again we find the convergence to be slow but fits as in (4) work well; furthermore, all the estimates of ω are similar, being in the $[0.4{:}0.6]$ interval. All these facts justify the claim that $\theta = 0$ whenever δE is quantized.

Just as in the continuous case, to analyze the *shape* of the distribution of δE we must choose a scale; the correct

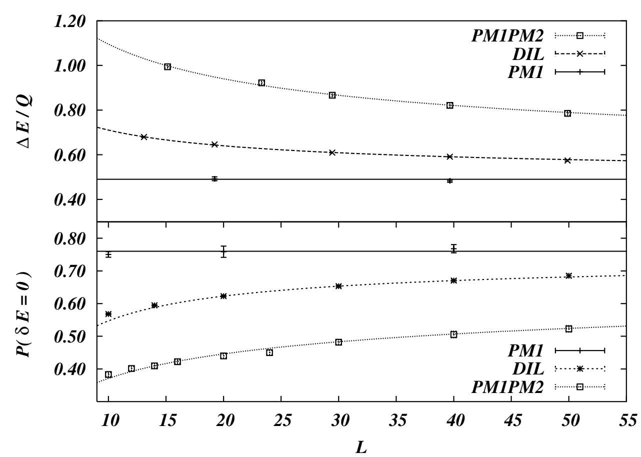

FIG. 2. $\Delta E/Q$ (top) and $P(\delta E = 0)$ (bottom) as a function of the system size for three discrete J_{ij} distributions: ± 1 (PM1), diluted ± 1 (DIL), and ± 1, ± 2 (PM1PM2).

choice is to compare the histograms after measuring all energies in units of the basic quantum Q. To test whether the histograms for the different J_{ij} distributions become identical in the large L limit, we plot in Fig. 2 (lower panel) the probability $P(\delta E = 0)$ to find a zero energy domain wall. The data suggest that the histograms become identical in the large L limit; i.e., they support universality. [Following Eq. (4), we fix the asymptotic value of $P(\delta E = 0)$ to be that given by the $J = \pm 1$ model, and then we determine ω; in the plot we show these fits; they are all good and the values of ω are close to 0.5.] We have checked in detail that this claim applies to the quantized distributions mentioned before *and* to the *DIL* model with 10% dilution.

Class 3: quantized energies revisited.—So far we have considered only situations with even values of L. If L is odd (and $J_{ij} = \pm 1$), the possible values of $\delta E/J$ are $\pm 2, \pm 6, \pm 10, \ldots$. The quantum Q is still the separation between the energy values, but the positions of the histogram entries are different (in particular, $\delta E = 0$ is not allowed). A somewhat trivial consequence of this is that necessarily $\theta \geq 0$ as $\Delta E/Q$ is greater or equal to $1/2$ for all L. Consider now the question of the universality of the histograms. We have checked within our error bars that the large L limit of $P(\delta E/Q)$ for the $J_{ij} = \pm 1$ model is the same as that obtained using the $J_2/J_1 = 3$ model (still with L odd, of course). This kind of quantization thus gives rise to a third class, again with $\theta = 0$.

Could there be further classes with quantized energies? Since we have imposed reflection symmetry of the distribution of the J_{ij}, the only possible histograms are the two we discussed: δE is a multiple of the quantum Q or of the form $(n + 1/2)Q$, where n is integer. If the universality class depends only on the possible histogram types, then no other classes arise.

Discrete does not mean quantized.—Let us also consider the case where the couplings are *discrete* but where there is no quantization. We consider the distribution $P(J) = \frac{1}{4}[\delta(J \pm J_1) + \delta(J \pm J_2)]$ (IRR for "irrational" hereafter), where $J_1 = 1$ and $J_2 = (1 + \sqrt{5})/2 \approx 1.618$ is the golden mean. Clearly, we have $\delta E = 2(nJ_1 + mJ_2)$, where n and m are integers. Since J_2/J_1 is irrational, the set of possible δE values becomes dense when $L \to \infty$ and so it is natural to conjecture that this $P(J_{ij})$ leads to domain wall energies in class 1. Our findings are that ΔE decreases with L and shows no sign of saturation, and a power law fit gives $\theta = -0.29 \pm 0.01$, the value associated with class 1. Our conjecture is thus substantiated by these findings.

The convergence of $P(\delta E/\Delta E)$ to its limit is more problematic: for finite L, the distribution is the sum of a finite number of delta functions: we can only hope to have a "weak convergence" to the P obtained with the Gaussian couplings. In these conditions it is appropriate to consider the integrated probability distribution $\Pi_<(X) \equiv \int_{-\infty}^{X} P(\delta E/\Delta E) d(\delta E/\Delta E)$. In Fig. 3 we plot $\Pi_<$ for the irrational and for some continuous cases. The plots are very similar, supporting the claim that the discrete distribution IRR leads to domain wall energies in class 1. We have also included $P(\delta E/\Delta E)$ as an inset in Fig. 3: we have used a small bin size that allows one to observe the complex structure.

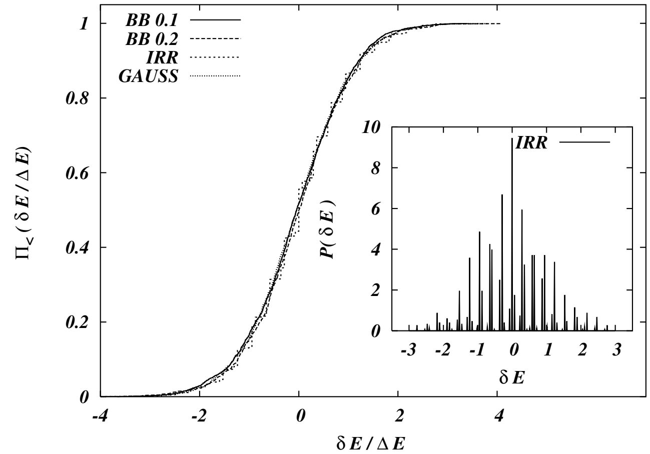

FIG. 3. The integrated probability distribution for $\delta E/\Delta E$ in the Gaussian case, in the two BB cases, and in the irrational J_2/J_1 case ($L = 60$). Inset: binned probability distribution of $\delta E/\Delta E$ for the irrational case (IRR), $L = 60$.

The case of hierarchical lattices.—The effect of having quantized δE can also be studied on hierarchical lattices. One advantage is that one can study very large sizes; a second is that one can access a *continuous* range of dimensions. We have focused on Migdal-Kadanoff lattices [6]; these are obtained by recursively "expanding" graphs. Starting with one edge connecting two sites, one replaces it by b paths in parallel, each composed of s edges in series, leading to bs new edges. This procedure is repeated hierarchically; after G "generations" the distance between the outermost spins is $L = s^G$, while the number of edges of the lattice is $(bs)^G$. The dimension of these lattices is $d = 1 + \ln(b)/\ln(s)$. One puts an Ising spin on each site and a coupling J_{ij} on each edge. Periodic boundary conditions simply imply that the two end spins must have the same value, and from this we define δE. The probability distribution of δE can be followed from G to $G + 1$. The recursion equations for $P(\delta E)$ make sense for any s [7]: s can be an integer but it can also be any positive real value. One may then compute θ for an interval of dimensions, using either continuous J_{ij} (for instance, to check universality [8,9]) or quantized J_{ij} couplings (our focus here).

In Fig. 4 we show θ as a function of dimension d (s is variable, b is fixed and set to 3). We show the values for continuous distributions and for when the quantization is of the form of class 3. As expected, if in one class $\theta > 0$, all classes lead to the same value of θ; i.e., quantization is irrelevant when the energy scale diverges. However, as soon as $\theta < 0$ in the continuous case, quantization gives

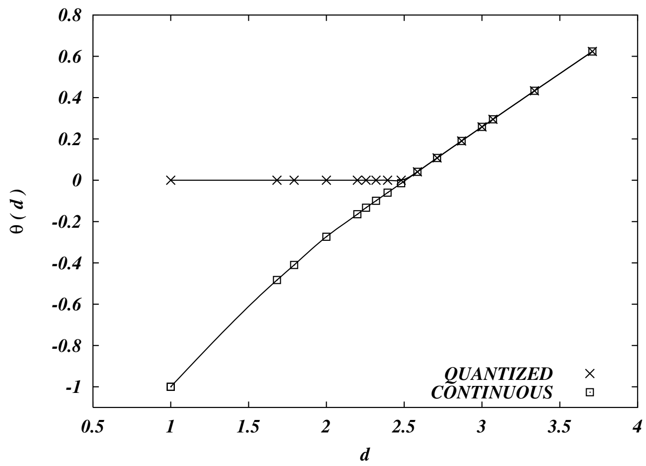

FIG. 4. θ as a function of d for a one-parameter family of Migdal-Kadanoff lattices; we display two sets of data points, one for continuous J_{ij} distributions, the other for quantized distributions.

rise to a histogram fixed point distribution in which the $|\delta E|$ are concentrated on the few lowest values and $\theta = 0$.

Similar results are obtained for class 2 quantization but there is an interesting difference. Indeed, since δE can be zero in this class, one sees two further fixed points. An obvious one is associated with having $P(\delta E = 0) = 1$; i.e., all domain wall energies vanish. It is easy to see that this fixed point is stable and has $\theta = -\infty$; there is no spin glass stiffness, and the system is paramagnetic even at zero temperature. The other fixed point is unstable and has $\theta = 0$. What is the interpretation of these two extra fixed points? To allow δE to be zero, one can think of the diluted model where some of the bonds have $J_{ij} = 0$. Clearly, when the dilution is strong enough, the nonzero bonds will no longer percolate and we are in a strongly paramagnetic phase; the renormalization group (RG) flow in this phase takes one to the $P(\delta E = 0) = 1$ fixed point. On the contrary, at low dilution, we are in a spin glass phase and the RG flows are towards the other stable fixed point. On the *boundary* of these two phases, the RG flows take one to another fixed point which is unstable: it is associated with the paramagnetic to spin glass transition as dilution is decreased. Such considerations have previously been developed for $d = 3$ Migdal-Kadanoff lattices [10].

Finally, we see that it is appropriate to define the lower critical dimension d_l^c from the end point of the $\theta = 0$ curve; $\theta = 0$ on its own does *not* signal $d = d_l^c$.

Discussion.—Our numerical evidence of universality [for both θ and $P(\delta E/\Delta E)$] is very strong for continuous and related distributions (see Figs. 1 and 3). But we also find universality classes when δE is quantized. This classification is substantiated by the behavior of θ and of the fixed point distributions of domain wall energies in Migdal-Kadanoff lattices. It is appropriate, however, to be cautious and to remark that the correction to scaling exponent ω we measure [see Eq. (4)] is small, $\omega \approx 0.5$. Because of that we are not able to completely exclude the Bray and Moore expectation that $\Delta E(L = \infty) = 0$ [11]. Our most extensive data are for the model DIL with $f = 0.2$. Here our fits give $\Delta E(L = \infty)/Q = 0.49(1)$ while if we force $\Delta E(L = \infty) = 0$, the χ^2 of the fit increases by 2.3; thus, $\Delta E(L = \infty) = 0$ is not excluded by our data though it appears as much less likely.

What is the source of the universality we observe? In the Migdal-Kadanoff lattices, the renormalization group transformation is clear and so the different classes are very natural. For the Euclidean lattices the existence of a renormalization group transformation for δE has not been established, but since our data point to universality, it should be possible to define such a transformation. Note that its fixed point [and thus $P(\delta E/\Delta E)$] will depend on the aspect ratio and on the fact that we use periodic boundary conditions. Our $P(\delta E)$ are thus *a priori* not comparable to those of [4], where one of the directions had free boundary conditions.

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