Superstatistical Mechanics of Tracer-Particle Motions in Turbulence

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The Lagrangian stochastic model of Reynolds [Phys. Fluids **15**, L1–4 (2003)] for the accelerations of fluid particles in turbulence is shown to predict precisely the observed Reynolds-number dependency of the distribution of Lagrangian accelerations and the exponents characterizing the observed extended self-similarity scaling of the Lagrangian velocity structure functions. Departures from superstatistics of the log-normal kind are accounted for and their impact upon model predictions is quantified.

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Recent experimental studies [1–5] have revealed that Lagrangian velocities and Lagrangian accelerations in high Reynolds-number turbulence are extremely intermittent quantities. Beck and co-workers [6–8] advocated the parametrization of Lagrangian acceleration statistics in terms of distributions that maximize nonextensive (Tsallis-like) entropies. Reynolds [9] subsequently developed a Lagrangian stochastic (LS) model for the simulation of fluid-particle accelerations that is exactly consistent with one such distribution, namely, a "superstatistical" distribution of the log-normal kind [8]. In contrast with more conventional modeling approaches [10], the new model incorporates explicitly fluctuations in the rate of dissipation of turbulent kinetic energy, ε. In this Letter it is shown that nearly precise agreement with the observed Reynolds-number dependency of the distribution of Lagrangian accelerations [2,3] is obtained when, in accordance with Kolmogorov's suggestion [11] and data from direct numerical simulation (DNS) [12], the variance of the logarithm of the dissipation rate, $\chi = \ln(\varepsilon/\langle\varepsilon\rangle)$ is approximated by $\sigma_\chi^2 = -0.354 + 0.289\ln R_\lambda$, where R_λ is the Reynolds number based upon the Eulerian Taylor microscale. The model is also shown to predict precisely the observed extended self-similarity scaling of the Lagrangian velocity structure functions [4] when, in accordance with the results of DNS [12], the temporal evolution of χ is modeled as an Ornstein-Uhlenbeck process.

The model of Reynolds [9] is prescribed by

$$dA = -\left(T_L^{-1} + t_\eta^{-1} - \sigma_{A|\varepsilon}^{-1}\frac{d\sigma_{A|\varepsilon}}{dt}\right)Adt - T_L^{-1}t_\eta^{-1}udt + \sqrt{2\sigma_U^2(T_L^{-1} + t_\eta^{-1})T_L^{-1}t_\eta^{-1}}d\xi, \quad (1)$$

where $A(t)$ is the Lagrangian acceleration of a fluid particle, $u(t)$ is its velocity at time t, and $d\xi$ is an incremental Weiner process with mean zero and variance dt. The two time scales, $T_L = 2\sigma_u^2/C_0\varepsilon$ and $t_\eta = 2a_0^*\nu^{1/2}/C_0\varepsilon^{1/2}$, are defined in terms of the *instantaneous* dissipation rate ε, two universal Lagrangian velocity structure constants, a_0^* and C_0, the kinematic viscosity ν, and the velocity variance σ_u^2. The conditional acceleration variance $\sigma_{A|\varepsilon}^2 = a_0^*\varepsilon^{3/2}\nu^{-1/2}$. By construction the model is exactly consistent with independent Gaussian distributions for accelerations and velocities with zero means and variances σ_u^2 and $\sigma_{A|\varepsilon}^2$. Model predictions for the distribution of Lagrangian accelerations are determined by $P(A) = \int P(A|\varepsilon)P(\varepsilon)d\varepsilon$.

Following Pope and Chen [13], the evolution of χ, along a fluid-particle trajectory, is prescribed by

$$d\chi = -(\chi - \langle\chi\rangle)T_\chi^{-1}dt + \sqrt{2\sigma_\chi^2 T_\chi^{-1}}d\xi', \quad (2)$$

where in accordance with DNS data [12], $\langle\chi\rangle = -1/2\sigma_\chi^2$ and $T_\chi = 2\sigma_u^2/C_0\langle\varepsilon\rangle$ and where $d\xi'$ is an incremental Weiner process (independent of $d\xi$) with mean zero and variance dt. The distribution of χ is Gaussian.

It is evident that Eqs. (1) and (2) can be combined into a single equation for the evolution of Lagrangian accelerations, which contains both multiplicative and additive noise and which is closely akin to the Langevin model of Eulerian intermittency developed by Laval *et al.* [14]. In both models the multiplicative noise results in intermittency.

According to Kolmogorov's (1941) hypothesis [15], the acceleration variance $\langle A^2\rangle = a_0\langle\varepsilon\rangle^{3/2}/\nu^{1/2}$, while the log-normal distribution of the dissipation rate implies that $\langle A^2\rangle = a_0^*\langle\varepsilon^{3/2}\rangle/\nu^{1/2}$, in accordance with Kolmogorov's refined hypothesis [15]. The universal constant a_0^* is related to a_0 by $a_0 = a_0^*\exp(3/8\sigma_\chi^2)$. Figure 1 shows that the observed dependency of a_0 on Reynolds number is captured well by this relationship when $a_0^* = 3.3$. Unless stated otherwise, model predictions presented in the remainder of this Letter were obtained for $a_0^* = 3.3$ and $C_0 = 7$. Figure 2 demonstrates that for $A/\sigma_A < 30$ the model is in startling good agreement with the experimental data [2,3,5]. The model is seen to predict accurately the increase in the extension of the tail of $P(A)$ with increasing Reynolds number. Note that this agreement is dependent only upon σ_χ and is independent of C_0, σ_u, and T_χ which determine the temporal evolution of Lagrangian accelerations and independent of a_0^* which determines the acceleration variance. For the extreme

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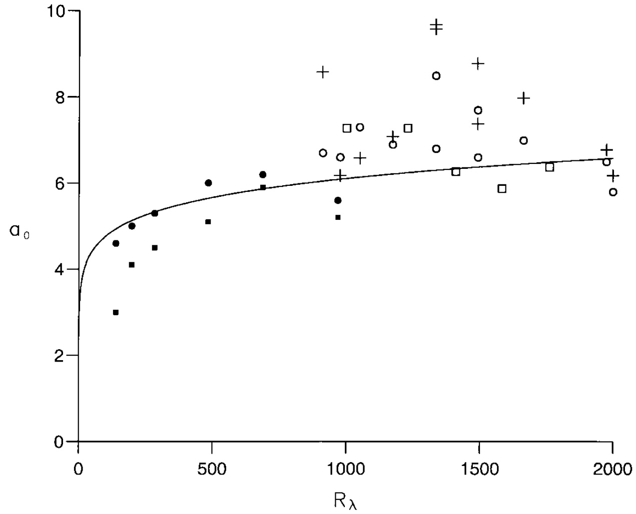

FIG. 1. Comparison of the predicted (solid line) and measured (symbols: ○, □, + [1]; ●, ■ [3]) dependence of the Kolmogorov constant a_0 on Reynolds number R_λ.

tails ($R_\lambda = 690$ data for $|A|/\sigma_A > 30$) the model tends to overestimate $P(A)$. This is further illustrated in Fig. 3, which shows that the model tends to overpredict the acceleration flatness. This is partly because the experimental data pertains to Lagrangian velocity increments at a small but finite time scale rather than to Lagrangian accelerations *per se*. To remove high frequency noise resulting from the electronics, mechanical vibrations and digital discretization, the data acquired for $R_\lambda = 690$ was filtered using a Gaussian kernel of width $0.17t_\eta$ [5]. The insets in Figs. 2 and 3 show that the tails and consequently the flatness of the predicted distribution of velocity increments over this time scale differ significantly from the model predictions for Lagrangian accelerations. The predicted distribution of velocity increments is seen to be in very close agreement with the experimental data and the predicted flatness, $F = 58$, coincides with the recent experimental estimate, $F = 55 \pm 4$ [5]. Crawford *et al.* [5] showed how the observed distribution of velocity increments, Δu, for $R_\lambda = 690$ can be parametrized accurately in terms of a stretched exponential with a superskewness, $S = \langle \Delta u^6 \rangle / \langle \Delta u^2 \rangle^3 \approx 4.5 \times 10^4$. The predicted superskewness of these velocity increments, $S = 4.5 \times 10^4$, is in exact agreement with the experimental estimate and is significantly less than the superskewness of the modeled accelerations, $S \approx 11 \times 10^5$.

The discrepancy between the measured and predicted distributions of Lagrangian accelerations may also be attributed, at least in part, to the assumption of a log-normal distribution for dissipation rates that is strictly applicable only for asymptotically large Reynolds numbers. The results of DNS indicate that departures from log-normality are significant at low Reynolds numbers ($R_\lambda \leq 93$) [12]. For example, the central moments of χ extracted from the results of DNS for isotropic turbulence with $R_\lambda = 38$, $\mu_3 = -0.24$, $\mu_4 = 3.22$, and $\mu_6 = 20.0$ are markedly different from the log-normal values of 0, 3, and 15 [12]. The impact of these departures from log-normality upon the modeled distribution of Lagrangian accelerations can be calculated by first constructing a distribution, $P(\chi)$, that is consistent with the DNS data for the central moments. The least biased choice for $P(\chi)$ and the one adopted here is

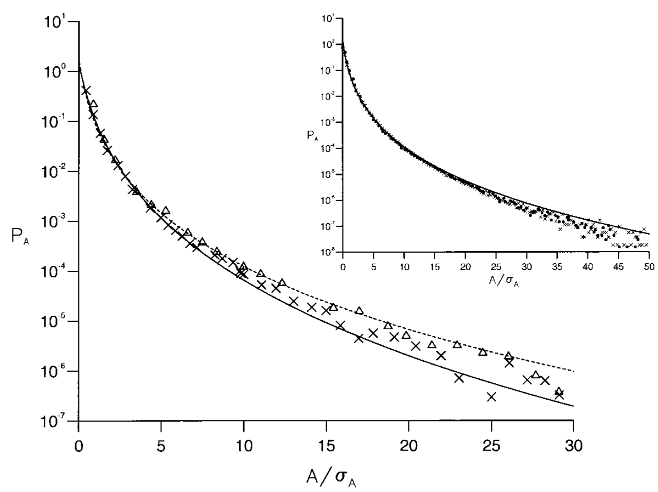

FIG. 2. Comparison of predicted (lines) and measured [2,3] distributions of tracer-particle accelerations for $R_\lambda = 200$ (×, solid line) and $R_\lambda = 970$ (△, dashed line). The inset shows a comparison between model predictions for the distribution of Lagrangian accelerations (solid line), model predictions for the distribution of velocity increments among 7.5×10^8 particles over a time scale $0.17t_\eta$ (×), and the experimental data [5] for $R_\lambda = 690$ (●).

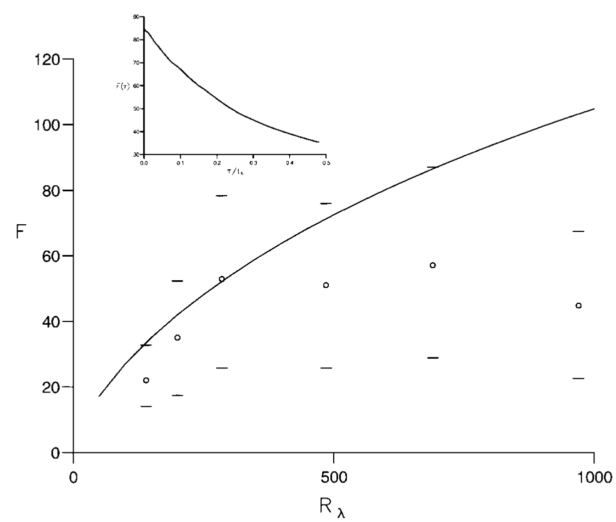

FIG. 3. Comparison of the predicted (solid line) and measured (symbols; error bars indicate experimental uncertainty) [3] flatness of the distribution of Lagrangian accelerations. Experimental values are lower bounds obtained for $|A|/\sigma_A \leq 30$ (more recent data for $R_\lambda = 690$ and $|A|/\sigma_A \leq 50$ yields $F = 55 \pm 4$ [5]). The inset shows the predicted flatness of velocity increments for $R_\lambda = 690$.

$$P(\chi) = \exp(c_0 + c_1\chi + c_2\chi^2 + c_3\chi^3 + c_4\chi^4 + c_6\chi^6). \tag{3}$$

This distribution maximizes the uncertainty about the missing information contained in fifth-, seventh-, and higher-order moments. The six coefficients, c_i, were determined from the conditions imposed by normalization and consistency with the DNS data for $\langle\chi\rangle$, $\langle\chi^2\rangle$, μ_3, μ_4, and μ_6. The distinction between (3) and the log-normal distribution together with the impact of the departures from log-normality upon the predicted distribution of accelerations are shown in Fig. 4. It is evident that departures from log-normality suppress the occurrence of extremely large but rare accelerations. They also reduce the occurrence of small accelerations ($|A|/\sigma_A \ll 1$) and so may account for the tendency of the log-normal model to overpredict the magnitude of the central core of $P(A)$.

Mordant *et al.* [4] measured incremental changes in particle velocities over finite times in high Reynolds-number turbulence having $R_\lambda = 740$. For times exceeding the Kolmogorov dissipative time scale, they observed that the Lagrangian velocity autocorrelation function, $R(\tau) = \langle u(t)u(\tau)\rangle/\sigma_u^2$, can be reproduced extremely precisely by an exponential, $R(\tau) = e^{-\tau/T}$, where T is a time scale associated with the "energy-containing" scales of motion. This key feature of turbulent dispersion is predicted by the model. For asymptotically large Reynolds numbers, the modeled form of the Lagrangian velocity autocorrelation function is given by

$$R(\tau) = \left\langle \exp\left[-\frac{C_0}{2\sigma_u^2}\int_0^\tau \varepsilon(t)dt\right]\right\rangle \overset{\tau \gg T_\chi}{\longrightarrow} \exp\left(-\frac{\tau}{T}\right), \tag{4}$$

where the angular brackets denote an ensemble average over dissipation histories and where $T = 2\sigma_u^2/C_0\langle\varepsilon\rangle$.

As illustrated in Fig. 5 for $R_\lambda = 740$, the modeled form of the Lagrangian velocity autocorrelation function at large times is also well described by an exponential when the Reynolds number is large but finite. The measured integral Lagrangian time scale [4] is predicted accurately by the model.

For the inertial subrange, Kolmogorov scaling dictates that the second-order Lagrangian velocity structure function $D_2^L(\tau) = \langle[u(t+\tau) - u(t)]^2\rangle = 2\sigma_u^2[1 - R(\tau)] = C_0\varepsilon\tau$. Mordant *et al.* [4] did not observe a plateau in $D_2^L(\tau)/\varepsilon\tau$ at C_0 but found that $D_2^L(\tau)/\varepsilon\tau$ reaches a maximum of $C_0^* \sim 2.9$ at $\tau = 20t_\eta$. Subsequent improvements in the estimate for ε yield a revised value of C_0^* of about 4.0 [16]. This data was acquired for particles having a size about 10 times larger than the Kolmogorov dissipation length scale and having a response time about twice as large as the Kolmogorov dissipation time scale. As a consequence, as compared to the case of fluid particles, the maximum in $D_2^L(\tau)/\varepsilon\tau$ occurs at later time and has a value that is lower than the actual value of C_0^*. Figure 5 (inset) demonstrates that model predictions for the true location and value of C_0^* are in accord with the experimental data when $C_0 = 6$. This value for C_0 is midway between the only two known high-quality independent measurements of C_0 [17].

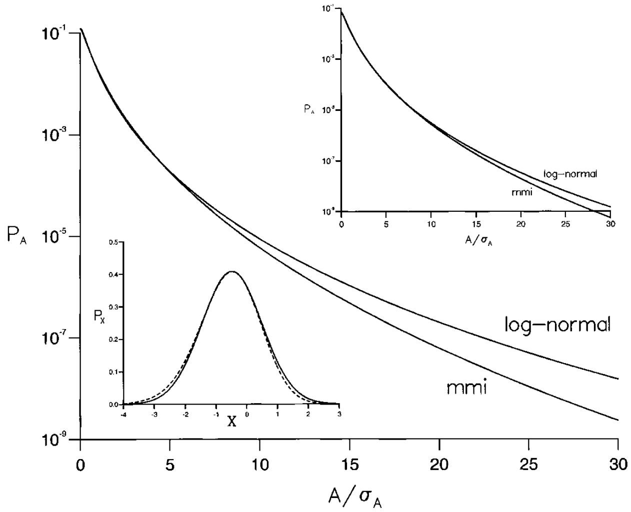

FIG. 4. Model predictions for the distribution of Lagrangian accelerations in isotropic turbulence with $R_\lambda = 93$ for a log-normal distribution (solid line, lower inset) of dissipation rates and for a maximum missing information (mmi) distribution (dashed line, lower inset) based upon DNS data [12]. Also shown (upper inset) are model predictions for $R_\lambda = 38$.

In accordance with the data of Mordant *et al.* [4], distributions of simulated velocity increments are almost Gaussian at integral time scales and progressively develop stretched exponential tails for small time increments. This continuous evolution with scale is quantified in Fig. 6 in terms of the excess flatness, $K = F - 3$. More generally, the evolution of these distributions can be characterized by the behavior of their moments (i.e., by the Lagrangian velocity structure functions), $D_q^L(\tau) = \langle u(t+\tau) - u(t)^q\rangle$. In the limit of very large Reynolds

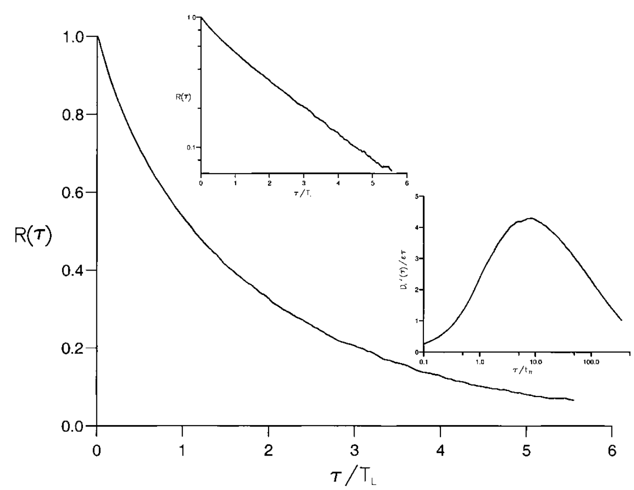

FIG. 5. Model predictions for the Lagrangian velocity autocorrelation function, $R(\tau)$, and the second-order Lagrangian velocity structure function, $D_2^L(\tau)$, nondimensionalized by the Kolmogorov scaling $\varepsilon\tau$ for $R_\lambda = 740$.

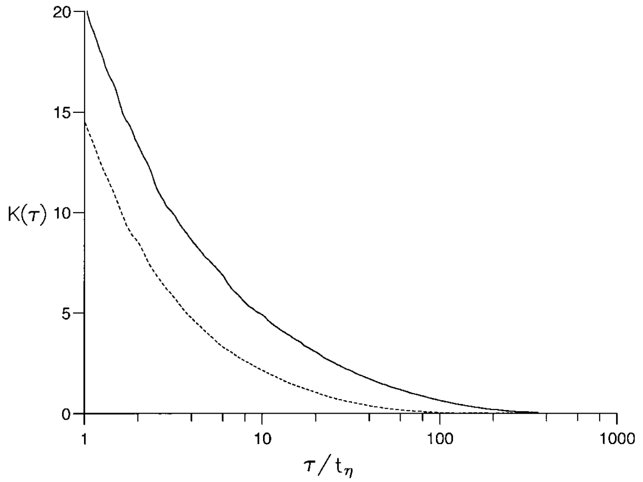

FIG. 6. The evolution of the excess flatness $K(\tau) = \langle(\Delta u)^4\rangle/\langle(\Delta u)^2\rangle^2 - 3$ of the predicted distribution of velocity increments Δu for $R_\lambda = 200$ (dashed line) and $R_\lambda = 740$ (solid line).

number, inertial subrange scaling dictates that $D_q^L(\tau) \propto \tau^{\xi_q}$. To compensate for the lack of a true inertial subrange at Reynolds numbers accessible to experiment, Mordant *et al.* [4] studied the relative scaling of the structure functions, $D_q^L(\tau) \propto D_2^L(\tau)^{\xi_q}$. Figure 7 demonstrates that such extended self-similarity is reproduced by the model. For $R_\lambda = 740$, the model predicts that $\xi_1 = 0.55$, $\xi_3 = 1.35$, $\xi_4 = 1.60$, and $\xi_5 = 1.76$ in precise accordance with the experimental values of $\xi_1 = 0.56 \pm 0.01$, $\xi_3 = 1.34 \pm 0.02$, $\xi_4 = 1.56 \pm 0.06$, and $\xi_5 = 1.8 \pm 0.2$. Consequently, the model is consistent not only with Kolmogorov's predictions for second-order quantities, but is also exactly consistent with the very strong intermittency observed at higher orders. The relative intermittency exponents were not found to be sensitively dependent upon either the Reynolds number or the integral time scale, T_χ, and for $R_\lambda > 75$ are essentially indistinguishable from the model predictions, $\xi_1 = 0.55$, $\xi_3 = 1.35$, $\xi_4 = 1.60$, and $\xi_5 = 1.75$, pertaining to asymptotically large Reynolds numbers. For asymptotically large Reynolds numbers, an exact model calculation yields $\xi_q = 1 + (q/2 - 1)D_{q/2}$, where $D_q \approx 1 - 0.2q$ is the spectral scaling function of dissipation. Mordant *et al.* [18] also introduced a LS model that can reproduce the observed Lagrangian intermittency. In that model, however, the intermittency exponents are essentially a model input, being incorporated into the model through a judicious choice for the driving noise. Gaussianity of the velocity increments is attained at time scales much larger than the integral time scale [16], and there is no explicit account of Reynolds-number effects. In accordance with experimental observations [18], both models predict that the acceleration is correlated over times comparable with the Kolmogorov dissipation time scale while the modulus of the acceleration, $|A|$, is correlated over times comparable with the integral Lagrangian time scale.

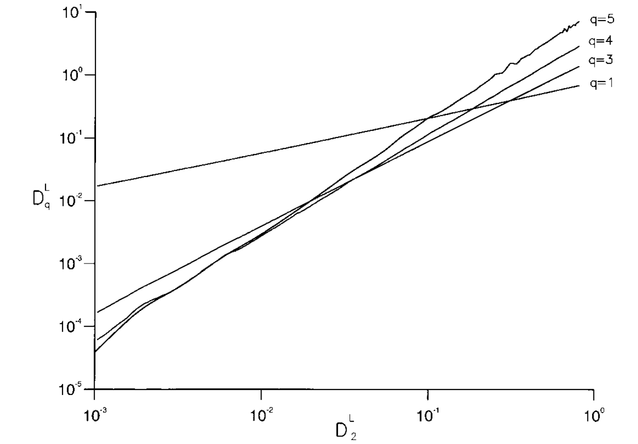

FIG. 7. Extended self-similarity plots of the Lagrangian velocity structure function variation. The dissipative range is not displayed.

In conclusion, a simple LS model for accelerations in turbulence has been presented that is in nearly precise agreement with key features of recently acquired experimental data for fluid-particle trajectories in high Reynolds-number turbulence.

This work has benefited from many stimulating discussions with Nicolas Mordant.

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