Large-*N* Reduction in the Continuum Three-Dimensional Yang-Mills Theory

R. Narayanan

Department of Physics, Florida International University, Miami, Florida 33199, USA

H. Neuberger*

School of Natural Sciences, Institute for Advanced Study, Princeton, New Jersey 08540, USA
(Received 26 March 2003; published 22 August 2003)

Numerical and theoretical evidence leads us to propose the following: Three-dimensional Euclidean Yang-Mills theory in the planar limit undergoes a phase transition on a torus of side $l = l_c$. For $l > l_c$ the planar limit is l independent, as expected of a noninteracting string theory. We expect the situation in four dimensions to be similar.

Introduction.—Yang-Mills theory in three dimensions is similar to Yang-Mills theory in four dimensions in exhibiting a positive mass gap, linear confinement, a finite temperature deconfinement transition, and a sensible large-*N* limit [1].

Non-Abelian gauge theories in three and four dimensions interact strongly at large distances and weakly at short distances. Doing calculations that bridge these two regimes remains a major challenge, of central importance to particle physics. It has been a long held hope that the task would simplify at infinite number of colors, *N*. Here, at strong coupling, a fitting hypothesis is that the theory describes free strings, while at weak coupling, the theory certainly describes weakly interacting particles. The result of this Letter indicates that strong and weak coupling are separated by a phase transition at infinite *N*. Specifically, we provide numerical evidence that SU(*N*) gauge theory on a three torus of side *l* undergoes a transition at a critical length, $l = l_c$. For any finite value of *N* there cannot be any phase transitions in this system. The existence of such a transition at infinite *N* is surprising, raises questions about the usually assumed smooth dependence of observables on momenta, and might indicate deeper connections between gauge theory, string theory and random matrix theories.

Over twenty years ago, in the context of SU(*N*) lattice gauge theory, Eguchi and Kawai [2] made the observation that at an infinite number of colors space-time can be replaced by a single point. This dramatic reduction in the number of degrees of freedom should make it easier to deal numerically with planar QCD than with ordinary, three color QCD. For a practical procedure it is essential that some version of large-*N* reduction also hold in the continuum, not just on the lattice. Previous attempts to define a continuous reduced model had problems with topology and fermions [3].

We focus on the continuum limit of pure lattice YM defined on a torus and try to establish that expectation values of traces of Wilson loop operators do not depend on the size of the torus. Wilson loops of arbitrary size can be folded up into the torus and correctly reproduced [4]. The lattice is essential because it provides a regularization with well defined loop equations [5]. Loop equations provide a convenient tool to establish reduction.

We restrict ourselves to three-dimensional theories for numerical reasons. We find that continuum large-*N* reduction holds so long as the torus is large enough. The critical side length of a symmetrical torus is denoted by l_c and is defined in terms of a microscopic fundamental physical scale of the theory. Solving the theory for some $l > l_c$ would produce complete and exact information at leading order in *N* for any *l*. The system as a whole undergoes a phase transition at $l = l_c$. The number of sites in a numerical simulation in a given direction, *L*, determines the maximal value the ultraviolet cutoff Λ can take. It is $\Lambda = (L/l_c)$. For a Wilson action the lowest *L* that has some semblance to continuum is $L = 3$. Thus, at the expense of larger *N* one can get numerically close to continuum using very small lattices. The values of *N* needed are of the order of 20 to 50 and this trade-off is worth taking.

If a similar result holds in four dimensions, a shortcut to the planar limit becomes a realistic option. Our experience makes us hopeful and our tools should allow us to tackle four dimensions in the future.

A lattice argument.—There is a global $Z^d(N)$ [$U^d(1)$ in the $N \rightarrow \infty$ limit] symmetry on the torus that leaves contractible Wilson loop operators invariant but multiplies Polyakov loops winding around a direction μ by a phase $e^{(2\pi\iota/N)k_\mu}$. The preservation of this symmetry is crucial for large-*N* reduction [6]. Eguchi and Kawai have shown that the lattice loop equations in the $N = \infty$ limit on a single site lattice are the same as on an infinite four-dimensional lattice as long as the $U^4(1)$ symmetry is unbroken. The continuum limit in the single site lattice model has to be taken by sending the lattice coupling $b = (1/g^2 N)$ to infinity, but in $d > 2$ a phase transition occurs, blocking the way. At the transition the $Z^d(N)$ symmetry breaks spontaneously, ruining the equivalence of loop equations. It is possible to fix the single site lattice model by quenching [3,6] or twisting [7] the system. We take a
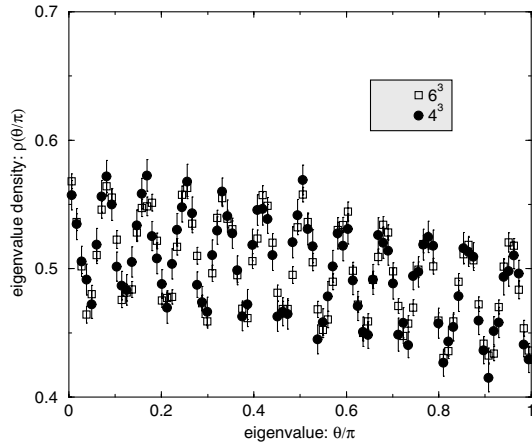
FIG. 1. Eigenvalue density distribution of a 4×4 Wilson loop on 4^3 (folded) and 6^3 (unfolded) at $b = 0.66$ and $N = 23$.

different approach here. The proof of Eguchi and Kawai goes through for a lattice torus of size $l_1 \times l_2 \ldots$ with arbitrary l_μ and in any dimension. The loop equations, together with boundary conditions for small loops, establish equality of expectation values of traces of operators associated with arbitrary finite closed loops in the infinite volume theory and their folded, contractible, counterparts on the torus. Suppose we reduced the model to only an L^d lattice with $L > 1$: Again we expect the global symmetry to break if $b > b_c(L)$ and reduction will hold for $b < b_c(L)$. $b_c(L)$ will increase with L and if $b_c(L)$ depends asymptotically on L as dictated by microscopic scaling for $d = 3, 4$ then continuum large-N reduction will hold if we take the limit by keeping $b < b_c(L)$ and taking $b \rightarrow \infty$.

In the approach pursued here, we have to deal with one lattice artifact. There will be a crossover in the lattice internal energy for the Wilson gauge action at some small b for a finite torus size and a finite N. The crossover becomes a "bulk" transition at infinite N, occurring at $b_c^B(N = \infty, L)$ for any finite lattice of size L^d in lattice units. Lattice large-N reduction would imply that $b_c^B(N = \infty, L)$ does not depend on L, $b_c^B(N = \infty, L) = b_c^{B\infty}$. This is consistent with numerical simulations. The loop equation, together with constraints which come from the parallel transporters being unitary matrices, produce the bulk transition without loosing their validity or changing their form. The lattice transition occurs when the unitary matrix associated with the one by one loop opens a gap at eigenvalue -1 in its spectrum in the large-N limit. As b increases further the gap widens. In the continuum this means that parallel transport around a tiny loop will not differ much(in norm) from the identity. Similar transitions occur at $b_c^B(N, L = \infty)$ for large enough N. The common limiting value at $b_c^B(N = \infty, L = \infty) = b_c^{B\infty}$ is rapidly approached. This family of transitions are lattice artifacts not associated with any symmetry breaking. Examples are the Gross-Witten [8]

transition in two dimensions and Creutz's transitions [9] for $N > 4$ in four dimensions.

Even though lattice reduction is valid on either side of $b_c^B(N = \infty, L)$ as long as one is below $b_c(L)$, we have to be above $b_c^B(N = \infty, L)$ to realize continuum reduction. For $L = 1$ (the Eguchi-Kawai model) and $d > 2$, the infinite N bulk and $Z^d(N)$ breaking transitions accidentally fuse at a $b_c \neq b_c^{B\infty}$. Similar "accidents" can happen for $L = 2, 3 \ldots$, but a window opens for large enough L between $b_c^{B\infty}$ and $b_c(L)$. In three dimensions, an $L = 3$ lattice already has a window.

$b_c^{B\infty} = 0.5$ [8] and $b_c(L) = \infty$ in $d = 2$. The $U^2(1)$ symmetries are not broken and continuum reduction works on tori of any size in two dimensions. The bulk transition occurs close to $b_c^{B\infty} = 0.4$ in $d = 3$. Ordinary scaling in $d = 3$ would require $L/b_c(L)$ to approach a finite nonzero limit as $L \rightarrow \infty$. Monte Carlo simulations were performed using a combination of heat-bath updates by SU(2) subgroups and of full SU(N) over-relaxation steps. Ultraviolet fluctuations in loop observables were suppressed by APE blocking [10]. We monitored the eigenvalue distribution of the Polyakov loops in the three directions and found that $0.6 < b_c(3) < 0.7$, $0.8 < b_c(4) < 0.9$, $1.0 < b_c(5) < 1.2$, and $1.2 < b_c(6) < 1.35$. When combined, these results indicate that the scaled critical coupling $L/b_c(L)$ is in the region $[4.2, 5]$. We compared folded and unfolded versions of the same loop on tori of different sizes and found the spectral densities associated with them to match as long as $[L/b(L)] \geq 5$. An example of such a comparision is shown in Fig. 1. We also checked scaling by comparing Wilson loops of the same physical size at different lattice spacings. An example of scaling is shown in Fig. 2.

Continuum perturbation theory.—If we had a scalar field theory where the field is a Hermitian $N \times N$ matrix we know that independence on the torus size is impossible. This dependence does not go away in the planar limit. On the level of Feynman diagrams (taken in
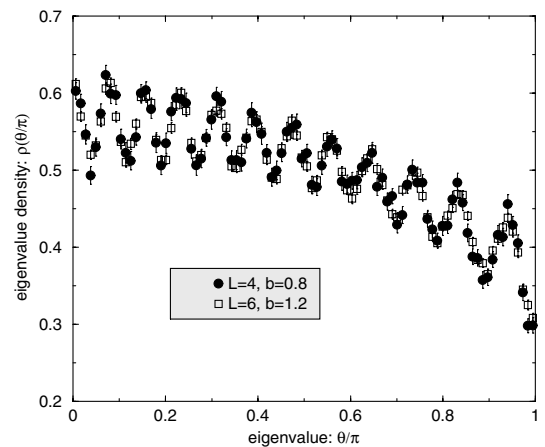

FIG. 2. Eigenvalue density distribution of $L \times L$ Wilson loop on L^3 for $L/b = 5$ and $L = 4, 6$. N is set to 23.

coordinate space) it is easy to calculate the dependence on torus size for large l, in particular, if the theory is massive [11]. The leading correction is exponentially suppressed in l and comes from one virtual particle going around a noncontractible circle on the torus. There is a stable particle like that and it is in the adjoint representation of SU(N). In the gauge case, if there is confinement, we could use only singlets under SU(N) and, at infinite N, these singlets make subleading contributions to the free energy at leading, $\mathcal{O}(N^2)$, order. We conclude that for a confining gauge theory a planar diagram with a ribbon (double-line) representation of propagators makes no contribution if one tears one of the propagators out of the surface and winds it around the torus.

Another way to see how reduction works in perturbation theory is to understand what happens to momentum space [6]. Having a torus means that momenta are quantized in units of $2\pi/l$ and there is no way around this for a massive scalar matrix field. In the gauge case the Feynman expansion starts from a constant gauge field background. The gauge invariant content of this moduli space consists of d sets of angles θ^i_μ which effectively fill the intervals between the quantized momenta making momentum space continuous and l independent. The filling has to be uniform and this is true at infinite N if the global $Z^d(N)$ symmetry is unbroken. The background—in a translation invariant gauge—is given by $A_\mu = \mathrm{diag}(\theta^1_\mu, \theta^2_\mu, \ldots, \theta^N_\mu)$ but only the set $\{e^{\iota l\theta^1_\mu}, e^{\iota l\theta^2_\mu}, \ldots, e^{\iota l\theta^N_\mu}\}$ labels truly distinct vacua. This is why $|\theta^i_\mu| < \frac{\pi}{l}$, exactly as needed to fill in the gaps between the $(2\pi k/l)$'s. At infinite N the vacuum is characterized by d eigenvalue distributions on the complex unit circle.

Hints from string theory.—In view of developments during the last few years [12] it seems more likely now than ever before that indeed large-N SU(N) pure gauge theories are equivalent to some string theory at zero string coupling. This means that the logarithm of the partition function defined on a finite torus and divided by the volume of the torus, is, in the planar limit, given by a sum of extended, spherical, two-dimensional excitations embedded in the same torus. But there is no way for the spherical surface to become noncontractible on the torus and thus it cannot detect that target space is a torus [13]. Hence, one can have no dependence on l. It is well known that simple string models on toroidal backgrounds cannot distinguish very large radii from very small ones; l_c, as a minimal radius, realizes a similar phenomenon in the unknown noninteracting string theory describing planar three-dimensional pure YM.

It used to be revolutionary to think that statistical field theories on finite volumes can have phase transitions. This is no longer true. To the early toy model examples [14] we can add now cases of true, full fledged field theories with real relativistic degrees of freedom, also developing phases transitions in the planar limit [12].

Large-N phase transitions.—Large-N transitions may emerge as quite ubiquitous in continuum gauge theories. There are transitions, such as the one presented in this Letter, that affect the system as a whole, but there are also other transitions that affect only a class of observables [15–18].

The basic observables used in our study have been the distribution of eigenvalues of Wilson and Polyakov loops. For large N these observables are unconventional because they involve traces of all powers of the basic unitary matrix, not only a few low powers. Thus, issues of renormalization require more work [19]. If these issues can be resolved, we might be able to exploit the fact that eigenvalues of large matrices have many universal properties [20]: The dynamics of the gauge theory could be encoded in the transformations one needs to carry out in order to bring these eigenvalue distributions to universal forms. While there are difficulties in continuum perturbation theory, the situation on the lattice is very clear: We numerically look for features that scale as the universal features of the field theory would have it.

The simplest strong-weak transition would be associated with Wilson loops: Small loops will have parallel transporters with a spectral gap and big loops will have almost uniform distributions to account for confinement in all finite irreducible representations. At finite N there are no gaps in the spectra but, in the range of the would be gaps, the eigenvalue density is exponentially suppressed as N increases.

Beyond the transition.—For l just a bit smaller than l_c exactly one of the $Z(N)$ factors in the $Z^3(N)$ breaks spontaneously. Thus, the 40 eight element cubic symmetry group of our equal sided torus breaks down to an eight-dimensional group acting in the plane perpendicular to the direction in which the Polyakov loop spectra took on nonuniform structure.

In order to prepare ourselves for what to look for when the torus is further squeezed we studied the 1^d EK model, now interpreted as a simple effective model for the dynamics of the vacuum manifold of the full system. Simulations we have carried out in three and four dimensions showed that at infinite N these models undergo a staircase of transitions, breaking one additional $Z(N)$ factor at a time. The possible continuum meaning of the various intermediate phases will have to wait for more work.

In supersymmetric YM gauge theories, compactified supersymmetrically on tori, the perturbative mechanism driving the spontaneous breaking of the $Z^d(N)$ symmetry can be eliminated. Beyond perturbation theory we do not know the answer, and other global symmetries come into play. It is conceivable that in some cases $l_c = 0$, indicative of a pure matrix model representation of the planar limit of a continuum gauge theory. Although the physical size is zero, regularization issues might require one to take $L \to \infty$ in a way correlated with $b \to \infty$, and the zero size

model may not admit a definition as the large-N limit of an ordinary matrix integral.

Future lattice work.—Building on earlier two-dimensional work we know how to calculate meson propagators in the planar limit. Meson momenta of values below the ultraviolet cutoff can be introduced by multiplying the original link matrices $U_\mu(x)$ by phase factors $e^{\iota p_\mu}$. The p_μ allow one to tune the momenta carried by the mesons to desired values. One has no finite volume effects to worry about: to get to the continuum limit one just increases b, making sure that l stays larger than l_c. Values of N in the range of few tens seem to be adequate. The lattice Dirac matrices are much smaller (and much denser) than in usual simulations. We would be able to address the smoothness of the two point meson correlation function, at infinite N, as a function of q^2, where q is Euclidean momentum. Could there be a nonanalyticity at some q^2? After all, if the crossover between *physical* strong and weak gauge forces happens in a range of scales that shrinks to zero at infinite N, phase transitions may occur in every observable, not only special ones, like Wilson loops. In four dimensions this could, finally, bring about a peaceful coexistence between large N and instantons [21].

In parallel simulations of the pure gauge case, large-N work will require a floating point effort per node that grows as N^3 while communication demands will only grow as N^2. So, PC farms with off the shelf communications would be well suited.

Conclusions.—Our final conjecture about three dimensions is stated in the abstract. We call it a "conjecture" because our numerical tests have been relatively modest and because the consequences of the conjecture could be far reaching: many applications of 't Hooft's large-N limit [22] assume analyticity in momenta and this assumption is now challenged. Our evidence is a combination of numerical work and more theoretical observations. On the numerical side we see the $Z^3(N)$ symmetry breaking point on the lattice change with lattice size in a way consistent with continuum scaling. Theoretically, lattice large-N reduction based on large-N loop equations is a strong coupling argument while averaging over the moduli space of constant Abelian connections at weak coupling resolves an apparent contradiction with conventional wisdom about finite size effects.

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*Permanent address: Rutgers University, Department of Physics and Astronomy, Piscataway, New Jersey 08855, USA.

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