Fluctuating Dimension in a Discrete Model for Quantum Gravity Based on the Spectral Principle

Luiz C. de Albuquerque,[1] Jorge L. deLyra,[2] and Paulo Teotonio-Sobrinho[2]

[1]*Faculdade de Tecnologia de São Paulo-DEG-CEETEPS-UNESP, Praça Fernando Prestes, 30,
01124-060 São Paulo, São Paulo, Brazil*

[2]*Universidade de São Paulo, Instituto de Física-DFMA, Caixa Postal 66318, 05315-970, São Paulo, São Paulo, Brazil*

The spectral principle of Connes and Chamseddine is used as a starting point to define a discrete model for Euclidean quantum gravity. Instead of summing over ordinary geometries, we consider the sum over generalized geometries where topology, metric, and dimension can fluctuate. The model describes the geometry of spaces with a countable number n of points, and is related to the Gaussian unitary ensemble of Hermitian matrices. We show that this simple model has two phases. The expectation value $\langle n \rangle$, the average number of points in the Universe, is finite in one phase and diverges in the other. We compute the critical point as well as the critical exponent of $\langle n \rangle$. Moreover, the space-time dimension δ is a dynamical observable in our model, and plays the role of an order parameter. The computation of $\langle \delta \rangle$ is discussed and an upper bound is found, $\langle \delta \rangle < 2$.

The quest for a quantum theory of gravity is one of the major goals in contemporary research. Any attempt in this direction involves the understanding of the space-time structure at a very short distance. It is generally believed that at the Planck scale space-time may not be described by a manifold [1]. The conventional geometrical setting of general relativity seems to be inadequate to describe the nonmanifold microstructures of space-time. Since the manifold structure has to appear at some macroscopic limit, it is natural to expect that one needs a generalization of ordinary geometry, such as noncommutative geometry (NCG), rather than a completely new formalism.

The basic idea coming from NCG [2] is that one can describe a Riemannian manifold $(M, g_{\mu\nu})$ in a purely algebraic way. There is no loss of information if, instead of the data $(M, g_{\mu\nu})$, one is given a triple $(\mathcal{A}, \mathcal{H}, D)$, where $\mathcal{A}$ is the C* algebra $C^0(M)$ of smooth functions on M, $\mathcal{H}$ is the Hilbert space of L^2 spinors on M, and $\mathcal{D}$ is the Dirac operator acting on $\mathcal{H}$. From the Gelfand-Naimark theorem it is known that the topological space M can be reconstructed from the set $\hat{\mathcal{A}}$ of irreducible representations of $C^0(M)$. Metric is also encoded, and the geodesic distance can be computed from $\mathcal{D}$. Here we will consider only commutative spectral triples — this is enough to go much beyond ordinary geometry. In particular, one can treat all Hausdorff topological spaces in this way. Given a pair $(M, g_{\mu\nu})$, one can promptly construct the corresponding triple $[C^0(M), L^2(M), \mathcal{D}]$. However, not all commutative spectral triples, or generalized geometries, come from a pair $(M, g_{\mu\nu})$. Nevertheless, one can always associate a Hausdorff space $M = \hat{\mathcal{A}}$ with a commutative spectral triple, where $\hat{\mathcal{A}}$ denotes the set of irreducible representations of $\mathcal{A}$. However, the space M may not be a manifold. Once we trade the original Riemannian geometry for its corresponding commutative triple we need a replacement for the Einstein-Hilbert action S_{EH}. The so-called spectral action of Chamseddine and Connes [3] is one possible candidate. It depends only on the eigenvalues of $\mathcal{D}$ and contains S_{EH} as a dominant term. In this paper, however, we shall use another spectral action.

The spectral action can be written for any triple, regardless of whether it comes from a manifold $(M, g_{\mu\nu})$ or not. In the spectral geometry approach it is conceivable to write the partition function

$$Z = \sum_{x \in \mathcal{X}} e^{-S[x]}, \tag{1}$$

where the "sum" is over the set $\mathcal{X}$ of all possible commutative spectral triples and S depends on the spectrum of $\mathcal{D}$. It includes all Hausdorff spaces and therefore all manifolds of all dimensions.

The framework of spectral NCG is appropriate to explore some difficult questions in quantum gravity. For example, Rovelli used a simple dynamical model based on a finite dimensional spectral triple to construct a quantum theory with a quantized physical distance between points [4]. In our model we look at the dimension as a dynamical quantity and study its expectation value.

The exact computation of (1) is a formidable task, not yet accomplished. However, the algebraic approach suggests ways of defining discrete approximations to the full theory. For instance, one may replace the algebra $\mathcal{A}$ by a finite dimensional algebra A_n. In this approach to discretization there is no need to introduce a lattice or simplicial decomposition of the underlying space. The approximation of $\mathcal{A}$ by a finite algebra works even if the spectral triple does not come from a manifold. Thus, it gives us a generalization of ordinary discretizations [5].

In this Letter we discretize (1) by sampling the set $\mathcal{X}$ with finite commutative spectral triples. We will think of

it as a useful toy model, which we believe captures some of the main features of the full one, Eq. (1).

The key role played by the eigenvalues of the Dirac (or Laplace) operator in the spectral action approach was emphasized in [6]. In our model they are also the natural dynamical variables due to the connection with random matrix theory (RMT). An important ingredient of the model is that the number of points can fluctuate. Moreover, in our simple model the space-time dimension is a dynamical observable and its expectation value can be computable from first principles.

Let us describe the ensemble $X \subset \mathcal{X}$ of geometries we will consider. A point of $x \in X$ is a commutative spectral triple $x = (\mathcal{A}, \mathcal{H}, D)$ where the commutative C* algebra $\mathcal{A}$ has a countable spectrum $\hat{\mathcal{A}}$. We divide X into subspaces X_n consisting of triples $(\mathcal{A}_n, \mathcal{H}_n, D)$ such that $\hat{\mathcal{A}}_n$ has a fixed number n of points. From the Gelfand-Naimark theorem it follows that elements of $\mathcal{A}_n$ are the (possibly infinite) sequences $a = (a_1, a_2, \ldots, a_n), a_j \in \mathbb{C}$. The Hilbert space $\mathcal{H}_n$ is given by vectors $v = (v_1, \ldots, v_n)$ with norm $||v||^2 \equiv \sum_{i=1}^{n} v_i^2 < \infty$. The elements of $\mathcal{A}$ are represented by diagonal matrices $\hat{a} = \mathrm{diag}(a_1, \ldots, a_n)$ acting on $\mathcal{H}_n$. Finally, the operator D is a $n \times n$ self-adjoint matrix. We will sample the space X by $X_1, X_2, \ldots, X_N$ and eventually take the limit $N \to \infty$.

Let L be a length scale such that the operator $\mathcal{D}$ given by $\mathcal{D} = D/L$ will be the analogue of the Dirac operator. The Chamseddine-Connes action depends on a cutoff function of the eigenvalues of D/L. The cutoff function is zero for eigenvalues of D greater than L and one otherwise [3,6]. In other words, the Boltzmann weight in Eq. (1) would be one outside a compact region in the eigenvalue space, leading to a divergent partition function [see Eq. (11)]. Let us consider a quadratic action instead:

$$S[x] = \mathrm{Tr}\left(\frac{\mathcal{D}}{\Lambda}\right)^2 \equiv \beta\,\mathrm{Tr}(D^2), \qquad (2)$$

where Λ is the inverse of Planck's length l_p, and $\beta = (l_p/L)^2$. Finally, we define the partition function $Z_N(\beta) = \sum_{n=0}^{N} z_n(\beta)$ where

$$z_n(\beta) = \int [dD] e^{-\beta\,\mathrm{Tr}(D^2)} \qquad (3)$$

is the partition function restricted to X_n, in other words, an integral over all independent matrix elements D_{ij}, where $[dD]$ is the usual measure for $n \times n$ Hermitian matrices [7]. The partition function $z_n(\beta)$ defines the one-matrix Gaussian unitary ensemble (GUE) [7]. A straightforward computation gives $z_n(\beta) = 2^{n/2}(\frac{\pi}{2\beta})^{n^2/2}$.

The expectation values of an observable $\mathcal{O}(D_{ij})$ restricted to X_n and for the entire ensemble are

$$\langle \mathcal{O} \rangle_{n,\beta} \equiv \int [dD]\mathcal{O}e^{-\beta\,\mathrm{Tr}(D^2)}/z_n(\beta), \qquad (4)$$

$$\langle \mathcal{O} \rangle(\beta) \equiv \sum_{n=1}^{N} P(n, \beta)\langle \mathcal{O} \rangle_{n,\beta}, \qquad (5)$$

respectively, where the function $P(n, \beta) = z_n(\beta)/\sum_n z_n(\beta)$ is interpreted as the probability of having a universe with n points. The simplest observable in our model is n, the number of points in $\hat{\mathcal{A}}$. By definition, n is constant in X_n, therefore $\langle n \rangle_{n,\beta} = n$. Thus we get

$$\langle n \rangle(\beta) = \frac{\sum_n n 2^{n/2}(\frac{\pi}{2\beta})^{n^2/2}}{\sum_n 2^{n/2}(\frac{\pi}{2\beta})^{n^2/2}}. \qquad (6)$$

The mean $\langle n \rangle$ ("average number of points in the universe") is not a continuous function of β at $\beta_c = \pi/2$, signaling the onset of a phase transition. Besides straightforward numerical calculation, there are other ways to show that the sum (6) converges for $\beta > \beta_c$ and diverges for $\beta < \beta_c$. Consider, for instance, the approximation of $Z_N(\beta)$ in the limit $N \to \infty$ by the first term of a Euler-Maclaurin expansion,

$$Z(\beta) = \lim_{N \to \infty} \sum_{n=0}^{N} z_n(\beta) = \int_0^\infty dx e^{-(1/2)a_\beta x^2 + bx} + \mathcal{R}(\beta), \qquad (7)$$

where $a_\beta = \ln(\beta/\beta_c)$, $b = \ln 2/2$. It is easily seen that

$$\langle n \rangle(\beta) = \frac{\partial}{\partial b}\,\ln Z(\beta), \qquad (8)$$

neglecting the remainder $\mathcal{R}$ in (7) [8]. Equation (7) suggests a nice interpretation. For $\beta > \beta_c$ ($\beta < \beta_c$) the quadratic term is positive (negative) and the integral converges (diverges). The phase transition at $\beta = \beta_c$ is triggered by the change of signal of a bilinear term in the "fields." The integral in (7) can be solved in the region $\beta > \beta_c$. After expanding a_β around the critical point, $a_\beta = (\beta - \beta_c)/\beta_c + \ldots$, we compute $\langle n \rangle$ by means of (8). The result is $\langle n \rangle(\beta) \sim (\frac{\beta-\beta_c}{\beta_c})^{-1} \equiv \tau^{-1}$. Hence, for $\beta > \beta_c$ the system is in a "finite" phase, characterized by a finite value of $\langle n \rangle$. As $\beta \to \beta_c^+$ $\langle n \rangle$ diverges with a (mean field) critical exponent $\nu = 1$. The rms deviation of $\langle n \rangle$ may be computed, with the result $\Delta n \equiv \sqrt{\langle n^2 \rangle - \langle n \rangle^2} \sim \tau^{-1/2}$. However, the relative width of the distribution, $\Delta n/\langle n \rangle$, decreases like $\tau^{1/2}$.

For $\beta < \beta_c$ the relevant universes have $\langle n \rangle = \infty$ and $\Delta n/\langle n \rangle = 0$. For a ∞-dimensional D one can define the dimension δ of the space $\hat{\mathcal{A}}$ from the eigenvalues of D. Let $\{\mu_0(D), \mu_1(D), \ldots\}$ be the modules of the eigenvalues (i.e., the singular values) of D organized in an increasing order. By the Weyl formula [2], the dimension δ is related to the asymptotic behavior of the eigenvalues for large k: $\mu_k(D) \approx k^{1/\delta}$. By definition $\delta = 0$ for finite dimensional spectral triples. We can argue that $\langle \delta \rangle$ is of the form

$$\langle \delta \rangle(\beta) = \begin{cases} f(\beta), & \text{if } \beta < \beta_c, \\ 0, & \text{if } \beta > \beta_c. \end{cases} \qquad (9)$$

This follows from the fact that for $\beta > \beta_c$ the probability $P(n, \beta)$ is localized around some finite n. Hence $\langle\delta\rangle$ works as an order parameter. The value $\beta_c = \pi/2$ separates $\langle\delta\rangle = 0$ from the rest.

In order to study the dimension we need to consider the spectral ζ function

$$\zeta(z) = \lim_{n\to\infty} \sum_{k=0}^{n} \mu_k^{-z} = \mathrm{Tr}\,(|D|^{-z}), \qquad (10)$$

where D is an ∞-dimensional matrix ($\mu_0 > 0$). The relation between the dimension and $\zeta(z)$ has been discussed in [9]. For large enough values of $\alpha = \mathrm{Re}(z)$, $\mathrm{Tr}(|D|^{-z})$ is well defined. One says that D has dimension spectrum Sd if a discrete subset $Sd = \{s_1, s_2, \ldots\} \subset \mathbb{C}$ exists, such that $\zeta(z)$ can be holomorphically extended to $\mathbb{C}/Sd$.

This definition is consistent with the Weyl formula. The set Sd has more than a single point when, for example, the geometry is the union of pieces of different dimensions [9]. In what follows we will look at an upper bound for the dimension: It may happen that $\mathrm{Tr}(|D|^{-\alpha}) = 0$ for large enough α, whereas for small values of α, $\mathrm{Tr}(|D|^{-\alpha}) = \infty$. Eventually, there is a value of α (say, α_c) for which $\mathrm{Tr}(|D|^{-\alpha_c})$ is finite and nonzero. The upper bound for the dimension will be $\delta = \alpha_c$.

In order to estimate $\langle\delta\rangle$ by means of (10), we rewrite (3) and (4) as integrals over the eigenvalues λ_k of D. The procedure is well-known [7], and leads to ($C_n \equiv \pi^{n(n-1)/2}/\prod_{k=1}^{n} k!$)

$$z_n(\beta) = C_n \int_{-\infty}^{\infty} [d^n\lambda]\Delta^2(\lambda_k)e^{-\beta\sum_{i=1}^{n}\lambda_i^2} \equiv C_n\Psi_{n,\beta}, \quad (11)$$

$$\langle\mathcal{O}(\lambda_i)\rangle_{n,\beta} = \int_{-\infty}^{\infty} [d^n\lambda]\mathcal{O}(\lambda_i)\left\{\frac{2^{n(n-1)/2}\beta^{n^2/2}}{\pi^{n/2}\prod_{k=1}^{n} k!}\Delta^2(\lambda_k)e^{-\beta\sum_{i=1}^{n}\lambda_i^2}\right\} \equiv \int_{-\infty}^{\infty} [d^n\lambda]\mathcal{O}(\lambda_i)\mathcal{P}_{n,\beta}(\lambda_k), \qquad (12)$$

where $\Delta(\lambda_k) = \prod_{i<j}(\lambda_j - \lambda_i)$ is the Vandermonde determinant (Jastrow factor), and $[d^n\lambda] \equiv \prod_{i=1}^{n} d\lambda_i$.

In RMT, $\Psi_{n,\beta}(\gamma)$ is interpreted as the positional partition function of an ensemble of equal charged particles (with positions given by λ_i) in two dimensions, moving along an infinite line, in thermodynamic equilibrium at temperature γ — the so-called "Dyson gas" [10]. Then, $\mathcal{P}_{n,\beta}(\lambda_1, \ldots, \lambda_n)$ defined in (12) is the probability of finding one particle at λ_1, one at λ_2, etc. The value of $\Psi_{n,\beta}(\gamma)$ is known from the Selberg's integral.

In the region $\beta \leq \beta_c$ the partition function $Z(\beta)$ is dominated by ∞-dimensional matrices. Thus, one may try to select the ∞-dimensional matrices out of the whole ensemble, and then compute the mean of the ζ functions following (10). However, from the standpoint of our statistical approach this procedure does not seem natural since the sum over n is a key ingredient in the whole construction. Hence, we look for a quantity related to the ζ function that captures the statistical nature of our model. Let us compute the mean value

$$\langle\mathrm{Tr}_\kappa|D|^{-\alpha}\rangle_{n,\beta} = \left\langle\sum_{k=1}^{n} |\lambda_k|^{-\alpha}\theta(|\lambda_k| - \kappa)\right\rangle_{n,\beta}. \qquad (13)$$

Bearing in mind (10), we want to study $\langle\mathrm{Tr}_\kappa|D|^{-\alpha}\rangle_{n,\beta}$ only in the region of eigenvalues where it is a decreasing function of α. Hence, we introduced an "infrared" cutoff κ, since there is a nonzero probability to find a configuration in the volume $\sum|\lambda_k| \leq \kappa$ around the origin. Using the symmetry of the integrand in (13) under permutations of the position indexes, we arrive at

$$\langle\mathrm{Tr}_\kappa|D|^{-\alpha}\rangle_{n,\beta} = \sqrt{\beta}\int_{\mathrm{out},\kappa} d\lambda_n|\lambda_n|^{-\alpha}\sigma_n(\sqrt{\beta}\lambda_n), \quad (14)$$

where we have used the definition of the spectral density $\sigma_n(\lambda) = \langle\sum\delta(\lambda - \lambda_k)\rangle_n$. Besides, $\int_{\mathrm{out},\kappa} \equiv \int_{-\infty}^{-\kappa} + \int_{\kappa}^{\infty}$.

The computation of (13) was reduced to a one-particle problem. It is known that $\sigma_n(\lambda) = \sum_k \phi_k^2(\lambda)$, $\phi_k(x)$ being the Weber-Hermite functions [7]. In the large n limit, $\sigma_n(\lambda)$ converges to a nonrandom function, Wigner's "semi-circle law,"

$$\sigma_n(\sqrt{\beta}\lambda) = \frac{\sqrt{2n}}{\pi}\sqrt{1 - \frac{\beta}{2n}\lambda^2}, \qquad (15)$$

for $|\lambda| < \mathcal{R}_{n,\beta} \equiv \sqrt{2n/\beta}$, and zero otherwise. The average eigenvalue density for Gaussian ensembles of Hermitian matrices with different values of n differ only by a change of scale, for large enough n (since $\mathcal{R}_{n,\beta} \sim \sqrt{n}$). We propose the following quantity in order to extract information on the ζ function (10):

$$\langle\mathrm{Tr}_{\kappa_{n,\beta}}|D|^{-\alpha}\rangle(\beta) = \lim_{N\to\infty}\sum_{n=1}^{N} P(n,\beta)\langle\mathrm{Tr}_{\kappa_{n,\beta}}|D|^{-\alpha}\rangle_{n,\beta}. \quad (16)$$

Hence, we are sampling the partial traces of the ensemble of ∞-dimensional Hermitian matrices by the total trace of finite dimensional matrices in the complete Gaussian ensemble of Hermitian matrices. Now, we study the asymptotic properties of $\langle\mathrm{Tr}_{\kappa_{n,\beta}}|D|^{-\alpha}\rangle_{n,\beta}$ as a function of α and n for $n \sim N$ large. In this case, one may use the semicircle law in (14),

$$\langle\mathrm{Tr}_{\kappa_{n,\beta}}|D|^{-\alpha}\rangle_{n,\beta} \approx \frac{2}{\pi}\sqrt{2n\beta}\int_{\kappa_{n,\beta}}^{\mathcal{R}_{n,\beta}} d\lambda_n|\lambda_n|^{-\alpha}\sigma_n(\sqrt{\beta}\lambda_n).$$

$$(17)$$

In (17) we are neglecting the contribution from the "tail" of $\sigma_n(\sqrt{\beta}\lambda)$, outside of the semicircle radius $\mathcal{R}_{n,\beta}$. There is an exponential decrease in the tail, so that the total number of particles (or eigenvalues) in this region is of order 2 [11]. This finite-size correction is not relevant to

our asymptotic analysis. Besides, the cutoff κ is a function of n and β, $\kappa_{n,\beta} = \epsilon\sqrt{2n/\beta}$. The mean particle (or level) spacing is $\overline{s}(\lambda, \beta) \sim \sigma_n^{-1}(\lambda, \beta)$, so that the mean spacing is almost uniform near the origin. Near the end points (edges) the spacing is highly nonuniform. This is the asymptotic region we are interested in. Our procedure is to select a slice of size $\Delta_{n,\beta,\kappa} = \mathcal{R}_{n,\beta} - \kappa_{n,\beta} = (1 - \epsilon)\mathcal{R}_{n,\beta}$ near the edge, so that the relative size of the slice, $\Delta_{n,\beta,\kappa}/\mathcal{R}_{n,\beta}$, does not depend on n and β. This choice ensures that we are treating matrices of different sizes n on the same footing. From (17) and (14) we obtain, after some manipulations,

$$\langle \mathrm{Tr}_{\kappa_{n,\beta}} |D|^{-\alpha} \rangle_{n,\beta} \approx \frac{2}{\pi}(2n)^{1-(\alpha/2)}\beta^{\alpha/2} \int_{\epsilon}^{1} \frac{dy}{y^{\alpha}}\sqrt{1 - y^2}. \tag{18}$$

The asymptotic behavior of $\langle \mathrm{Tr}_{\kappa_{n,\beta}}|D|^{-\alpha}\rangle_{n,\beta}$ does not depend in an essential way on the particular choice of ϵ, as long as we keep $\epsilon \neq 0$.

Now we use the asymptotic formula (18) in (16) and search for the value α_c for which, as $N \to \infty$ and $\beta \to \beta_c$, $\langle \mathrm{Tr}_{\kappa_{n,\beta}}|D|^{-\alpha}\rangle$ diverges (converges to zero) if $\alpha < \alpha_c$ ($\alpha > \alpha_c$), with $\langle \mathrm{Tr}_{\kappa_{n,\beta}}|D|^{-\alpha_c}\rangle$ finite and nonzero. This gives an upper bound for the dimension of the "condensed" manifold in the infinite phase ($\beta \leq \beta_c$), which is $\langle \delta \rangle < \alpha_c$. We obtain

$$\langle \mathrm{Tr}_{\kappa_{n,\beta}}|D|^{-\alpha}\rangle(\beta) \sim \lim_{N\to\infty} \sum_{n=1}^{N} P(n,\beta)n^{1-(\alpha/2)}. \tag{19}$$

In the finite phase ($\beta > \beta_c$) the sum in (19) converges for $\alpha \geq 0$. We conclude that $\alpha_c = 0$ (i.e., $\langle \delta \rangle = 0$) for $\beta > \beta_c$, as expected. From the behavior of $P(n,\beta)$ in the infinite phase it follows that the convergence of the sum in (19) is dictated by the behavior of $\Gamma_{n,\alpha} = n^{1-(\alpha/2)}$ in the limit $n \sim N \to \infty$. For $\beta \leq \beta_c$ we get $\Gamma_{n,\alpha} \to \infty$ if $\alpha < 2$, and $\Gamma_{n,\alpha} \to 0$ if $\alpha > 2$. For $\alpha = 2$ it turns out that $\langle \mathrm{Tr}_{\kappa_{n,\beta}}|D|^{-2}\rangle(\beta) \sim 1$. Therefore, we obtain the upper bound $\langle \delta \rangle < 2$.

Notice that we do not have a definition of the dimension as an operator [as suggested by (9)], from which it will be possible to compute its average in the ensemble of all Hermitian matrices, as we have do for n in (6). Instead, our guide is the operational definition encapsulated in (16). In order to go beyond the upper bound computed above we need to find a suitable observable which reduces to the classical dimension (as given by the Weyl formula) in the limit $n \to \infty$. This eventually may lead to a numerical computation of $\langle \delta \rangle$. Besides, one sees that the reason for the upper bound $\langle \delta \rangle < 2$ lies in the semicircle law and its leading $\sqrt{\lambda}$ behavior near the edge. It is known from 2D models of discretized pure quantum gravity [12,13] that some special matrix poly-

nomial potentials $V(D) = \sum a_k \mathrm{Tr} D^{2k}$ may lead, in a suitable scaling limit, to a behavior near the edges different from the square root. Thus, a possible way to obtain a bound of higher dimension would be to include higher polynomial interactions, or modify the quadratic one in (2) including more (internal) symmetries besides the unitary one. These and other questions are under study, and will be reported elsewhere.

To conclude, in this Letter we proposed a discrete model for quantum gravity based on the spectral principle. The model is connected with the GUE of Hermitian matrices, and contains the mean number of points, $\langle n \rangle$, and the dimension of the space-time, $\langle \delta \rangle$, as dynamical observables. We have shown that the model has two phases: a finite phase with a finite value of $\langle n \rangle$ and $\langle \delta \rangle = 0$, and an infinite phase with a diverging $\langle n \rangle$ and a finite $\langle \delta \rangle \neq 0$. The critical point was computed, $\beta_c = \pi/2$, as well as the critical exponent of $\langle n \rangle$. Moreover, an upper bound for the order parameter $\langle \delta \rangle$ was found, $\langle \delta \rangle < 2$.

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