Phase Modulated Thermal Conductance of Josephson Weak Links

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We present a theory for quasiparticle heat transport through superconducting weak links. The thermal conductance depends on the phase difference (ϕ) of the superconducting leads. Branch-conversion processes, low-energy Andreev bound states near the contact, and the suppression of the local density of states near the gap edge are related to phase-sensitive transport processes. Theoretical results for the influence of junction transparency, temperature, and disorder, on the conductance, are reported. For high-transmission weak links, $D \to 1$, the formation of an Andreev bound state leads to suppression of the density of states for the continuum excitations, and thus, to a reduction in the conductance for $\phi \simeq \pi$. For low-transmission ($D \ll 1$) barriers resonant scattering leads to an increase in the thermal conductance as T drops below T_c (for phase differences near $\phi = \pi$).

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The Josephson effect in superconducting weak links is perhaps the best known example of macroscopic phase coherence. In addition to the superconducting tunneling current, $j_s = j_c \sin\phi$, Josephson [1] showed that the electrical current through a tunnel junction includes Ohmic terms, $j_d = (\sigma_0 + \sigma_1 \cos\phi)V$, where ϕ is the phase difference between the two superconductors, and V is the voltage across the junction. These terms describe dissipative quasiparticle tunneling when the junction is biased by a voltage. The term, $\sigma_1 V \cos\phi$, is attributed to "interference" between Cooper-pair and quasiparticle tunneling [1–3], and averages to zero in voltage-biased junctions.

Phase-modulated dissipative currents are characteristic of any type of superconducting weak link. For example, the thermal current through a temperature-biased superconductor-insulator-superconductor (SIS) junction is predicted to be a periodic function of ϕ [4]. For a stationary phase difference the thermal conductance is also stationary. Thus, in contrast to voltage-biased junctions, the phase-modulated thermal conductance does not average to zero. However, less is known about phase-sensitive thermal transport across superconducting weak links compared with their current-voltage characteristics. Recent investigations of heat transport through SIS junctions are based on the tunnel Hamiltonian method [5], while Kulik and Omel'yanchuk [6] calculated the thermal current for the opposite extreme of a perfectly transmitting superconducting constriction (a "pinhole"). In terms of the transmission coefficient of the interface potential barrier between two superconductors, the SIS junction corresponds to $D \ll 1$, while the pinhole corresponds to perfect transmission, $D \to 1$.

In the following we present a theory for quasiparticle heat transport through superconducting point contacts for any junction transparency, and as a function of temperature and disorder. The thermal conductance is sensitive to spatial inhomogeneities of the order parameter, particularly changes in phase, which lead to branch conversion between particlelike and holelike excitations [7], and to the formation of low-energy bound states in the vicinity of the point contact. The bound state spectrum and transmission probabilities for continuum excitations are strongly modified by the junction transparency, which leads to large changes in the thermal conductance of the junction.

To study quasiparticle transport through temperature-biased superconducting weak links, we use the method of nonequilibrium quasiclassical Green functions [8]. In this formalism the advanced and retarded Green functions, $\hat{\mathfrak{G}}^{A,R}$, describe the local spectrum of excitations for the system, while the Keldysh Green functions, $\hat{\mathfrak{G}}^{K}$, carry the information about the nonequilibrium population of these states. Each propagator, $\hat{\mathfrak{G}}^{A,R,K}(\mathbf{p}_f, \epsilon; \mathbf{R}, t)$ is a 2×2 matrix in particle-hole (Nambu) space obeying transportlike equations for excitations of energy ϵ moving along classical trajectories labeled by the Fermi momentum, $\mathbf{p}_f$. We use the notation of Ref. [8], set $\hbar = 1$, $k_B = 1$, and consider spin-independent transport in spin-singlet superconductors.

The basic model for superconducting weak links considered here is that of a constriction of diameter and length on the nanometer scale, much smaller than the coherence length, $\xi_\Delta = v_f/\pi\Delta$, and the bulk elastic and inelastic mean-free paths (see Fig. 1). The potential barrier located at $z = 0$ is characterized by a transmission (or reflection) probability, $D(\mathbf{p}_f)$ [or $R(\mathbf{p}_f) = 1 - D(\mathbf{p}_f)$], for normal-state quasiparticles with Fermi momentum $\mathbf{p}_f$ incident on the interface. The coupling between the two superconductors, S_1 and S_2, can then be described by a boundary condition connecting the Green functions for the two superconductors at the junction interface [9]. The order parameter at the junction interface for the superconductor S_j is $\Delta\, e^{i\phi_j}$, and the temperature at the junction

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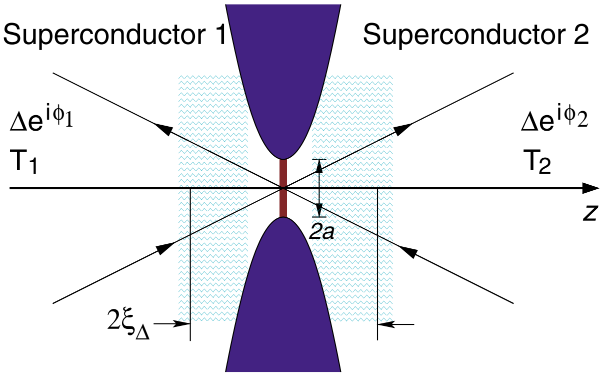

FIG. 1 (color online). Temperature-biased ScS weak link geometry. Quasiparticle trajectories are coupled by reflection and transmission at $z = 0$. The shaded boundaries define the region ($\sim \xi_\Delta \gg a$) where superconductivity and the excitation spectrum are strongly modified for $\phi_1 \neq \phi_2$.

interface for superconductor S_j is T_j for $j = 1, 2$. The phase difference is denoted by $\phi = \phi_2 - \phi_1$, and the temperature bias is $\delta T = T_2 - T_1$.

Recently, Eschrig [8] recast Zaitsev's boundary condition into a convenient form by parametrizing the quasiclassical Green functions and transport equations using Shelankov's projection operators [10] and generalized spectral, $\gamma^{R,A}$, and distribution, x^K, functions obeying Riccatti-type transport equations. We use the Riccatti formulation of the quasiclassical equations with the boundary condition of Ref. [8] to solve for the Keldysh Green's functions and heat current for superconducting weak links driven out of equilibrium by a temperature bias. The propagators for trajectories incident, $\hat{\mathfrak{G}}_+^{R,A,K} = \hat{\mathfrak{G}}_1^{R,A,K}(\mathbf{p}_f \cdot \hat{\mathbf{z}} > 0, z = 0^-)$, and reflected, $\hat{\mathfrak{G}}_-^{R,A,K} = \hat{\mathfrak{G}}_1^{R,A,K}(\mathbf{p}_f \cdot \hat{\mathbf{z}} < 0, z = 0^-)$, by the interface from the S_1 side can be used to determine the heat current through the interface,

$$I_\epsilon = \mathcal{A} N_f \int \frac{d\epsilon}{4\pi i} \epsilon v_f \langle \mathrm{Tr}[\hat{\mathfrak{G}}_+^K(\mathbf{p}_f, \epsilon) - \hat{\mathfrak{G}}_-^K(\mathbf{p}_f, \epsilon)] \rangle, \quad (1)$$

where N_f is the normal-state density of states at Fermi-surface, $\mathbf{v}_f$ is the Fermi velocity and $\mathcal{A} = \pi a^2$ is the cross-sectional area of the constriction. The Fermi-surface average includes the direction cosine for projection of the group velocity along the direction normal to the interface; i.e. $\langle \cdots \rangle = \frac{1}{2} \int_0^{\pi/2} d\theta \sin\theta \cos\theta (\cdots)$, where $\theta = \arccos(\hat{\mathbf{v}}_f \cdot \hat{\mathbf{z}})$. We consider the case in which S_1 and S_2 are identical s-wave superconductors with isotropic Fermi surfaces; it is straightforward to generalize the results to two different s-wave superconductors. We also consider small junctions, $a \ll \xi_\Delta$; in this limit the Josephson supercurrent through the contact is small by at least a factor a/ξ_Δ compared to the bulk critical current, so pair-breaking corrections to the order parameter by the supercurrent can be neglected [11]. Thus, to a good approximation the order parameters take their bulk equilibrium values, but with local values for the phase. Also for small constrictions the thermal resistance of the junction is much larger than in that in the bulk, so the temperature drop occurs essentially at the junction.

With these considerations, $\hat{\mathfrak{G}}_\pm^K$ can be calculated as follows. First, we express the Green function in terms of Riccati amplitudes [8]. For example,

$$\hat{\mathfrak{G}}_+^K = \frac{-2\pi i}{N_1^R N_1^A} \begin{pmatrix} x_1^K + \tilde{X}_1^K \gamma_1^R \tilde{\gamma}_1^A & x_1^K \Gamma_1^A - \tilde{X}_1^K \gamma_1^R \\ x_1^K \tilde{\Gamma}_1^R - \tilde{X}_1^K \tilde{\gamma}_1^A & \tilde{X}_1^K + x_1^K \tilde{\Gamma}_1^R \Gamma_1^A \end{pmatrix}, \quad (2)$$

where $N_1^R N_1^A = (1 + \gamma_1^R \tilde{\Gamma}_1^R)(1 + \tilde{\gamma}_1^A \Gamma_1^A)$. The advanced amplitudes are obtained from retarded functions using the symmetry $\gamma^A = -(\tilde{\gamma}^R)^*$. In the point contact geometry, the amplitudes and distribution functions for incoming quantities (lower case) take their local equilibrium values: $\gamma_j^R = \tilde{\gamma}_j^R(-\epsilon)^* = -i\Delta\, e^{i\phi_j}/[\varepsilon^R + i\sqrt{\Delta^2 - (\varepsilon^R)^2}]$, $x_j^K = \tilde{x}_j^K(-\epsilon)^* = (1 - |\gamma_j^R|^2)\tanh(\epsilon/2T_j)$, for $j = 1, 2$. Using the boundary conditions of Ref. [8], we construct the corresponding functions for outgoing trajectories (upper case),

$$\tilde{\Gamma}_1 = \frac{R(1 + \tilde{\gamma}_2\gamma_2)\tilde{\gamma}_1 + D(1 + \tilde{\gamma}_1\gamma_2)\tilde{\gamma}_2}{1 + R\gamma_2\tilde{\gamma}_2 + D\tilde{\gamma}_1\gamma_2}, \quad (3)$$

$$\tilde{X}_1 = \frac{R|1 + \tilde{\gamma}_2\gamma_2|^2\tilde{x}_1 + D|1 + \tilde{\gamma}_1\gamma_2|^2\tilde{x}_2 - RD|\tilde{\gamma}_1 - \tilde{\gamma}_2|^2 x_2}{|1 + R\gamma_2\tilde{\gamma}_2 + D\tilde{\gamma}_1\gamma_2|^2},$$

where we omitted the superscripts. Inserting these expressions into Eq. (2), and performing the analogous calculation for $\hat{\mathfrak{G}}_-^K$, we obtain explicit expressions for the phase-sensitive thermal conductance, defined by $I_\epsilon = -\kappa(\phi, T)\delta T$, in the limit $\delta T \to 0$. The general result can also be used to calculate the heat current beyond the linear response limit. A more detailed derivation and discussion will be presented elsewhere [12].

In the clean limit, $\varepsilon^R \to \epsilon + i0^+$, the thermal conductance is expressed in terms of the transmission of bulk excitations of energy $\epsilon \geq \Delta$ and group velocity, $v_g(\epsilon) = v_f\sqrt{\epsilon^2 - \Delta^2}/\epsilon$ through the junction,

$$\kappa(\phi, T) = \mathcal{A} \int_\Delta^\infty d\epsilon \mathcal{N}(\epsilon)[\epsilon v_g(\epsilon)]\mathfrak{D}(\epsilon, \phi)\left(\frac{\partial f}{\partial T}\right), \quad (4)$$

where $f(\epsilon) = (e^{\epsilon/T} + 1)^{-1}$ is the Fermi function, $\mathcal{N}(\epsilon) = N_f\epsilon/\sqrt{\epsilon^2 - \Delta^2}$ is the bulk density of states (DOS), and $\mathfrak{D}(\epsilon, \phi) = \mathfrak{D}_{\mathrm{ee}}(\epsilon, \phi) + \mathfrak{D}_{\mathrm{eh}}(\epsilon, \phi)$ is the transmission coefficient for the heat current, which is the sum of

$$\mathfrak{D}_{ee}(\epsilon, \phi) = D\frac{(\epsilon^2 - \Delta^2)(\epsilon^2 - \Delta^2\cos^2\frac{\phi}{2})}{[\epsilon^2 - \Delta^2(1 - D\sin^2\frac{\phi}{2})]^2}, \quad (5)$$

the transmission coefficient for electronlike (holelike) quasiparticles remaining electronlike (holelike), and

$$\mathfrak{D}_{eh}(\epsilon, \phi) = D(1-D)\frac{(\epsilon^2-\Delta^2)\,\Delta^2\sin^2\frac{\phi}{2}}{[\epsilon^2-\Delta^2(1-D\sin^2\frac{\phi}{2})]^2}, \quad (6)$$

the transmission coefficient for electronlike (holelike) quasiparticles with branch conversion to holelike (electronlike) quasiparticles. Here we neglect the angular dependence of the barrier transmission and reflection probabilities [13].

The heat current density carried by bulk quasiparticles of energy ϵ in the superconducting leads reduces to the normal-state current density, $\mathcal{N}(\epsilon)[\epsilon v_g(\epsilon)] = N_f \epsilon v_f$; the increase in bulk DOS is compensated by the reduction in the group velocity. For $\phi = 0$, $\mathfrak{D}(\epsilon, \phi)$, reduces to the barrier transmission probability for normal-state quasiparticles, $\mathfrak{D}(\epsilon, 0) = D$. Thus, for $\phi = 0$ the thermal conductance of the ScS contact is simply reduced by the opening of the gap in the quasiparticle spectrum. However, for $\phi \neq 0$ the transmission coefficient, $\mathfrak{D}(\epsilon, \phi)$, includes the modification of the local DOS near the contact by the formation of an Andreev bound state (ABS) with energy below the gap edge, as well as the particle-hole coherence amplitudes which alter direct (*ee*) transmission and generate branch-conversion (*eh*) scattering.

The relative importance of the direct (*ee*) and branch-conversion (*eh*) processes to the phase dependence of the thermal conductance depends on the barrier transparency D. For both processes the ABS plays a central role in controlling the phase modulation of the conductance. The bound state leads to a reduction in the local DOS near the contact and to a corresponding suppression of the transmission coefficient for excitations with $\epsilon = \Delta$. For moderate to high-transmission barriers ($D \gtrsim 0.5$) this leads to suppression of the thermal conductance for $\phi \approx \pi$. For low-transmission barriers ($D \ll 1$) multiple Andreev reflection leads to a shallow bound state just below the continuum edge at $\epsilon_b = \Delta\sqrt{1 - D\sin^2(\frac{\phi}{2})}$. The spectral weight of the ABS is derived from the continuum states near $\epsilon = \Delta$, which suppresses the divergence at $\epsilon = \Delta$ at the cost of a large, but finite, resonant enhancement in the transmission of quasiparticles at energies, $\epsilon \approx \Delta[1 + \frac{1}{2}D\sin^2(\frac{\phi}{2})]$, *above* the gap [Fig. 2(c)]. The resonance generates a strong enhancement of the thermal conductance as the phase is tuned to $\phi = \pi$. These features, as well as the evolution of the phase modulation of the conductance with barrier transmission, are shown in Fig. 2, where we plot the normalized conductance, $\kappa(\phi)/\kappa(\phi = 0)$ for $0 < D \leq 1$ and $T = 0.5\Delta$ ($T = 0.72\,T_c$). For intermediate values of the barrier transparency, $D \simeq 0.5$, the phase dependence of the conductance is a nonmonotonic function of ϕ for $0 \leq \phi \leq \pi$ [Fig. 2(b)], although the amplitude of these oscillations is very small. For $D = 1$, transmission with branch conversion drops out ($\mathfrak{D}_{eh} = 0$) and the transmission coefficient for the quasiparticle heat current reduces to $\mathfrak{D}(\epsilon, \phi) = (\epsilon^2 - \Delta^2)/[\epsilon^2 - \Delta^2\cos^2(\frac{\phi}{2})]$, with a resulting thermal conductance in agreement with Ref. [6] for a pinhole.

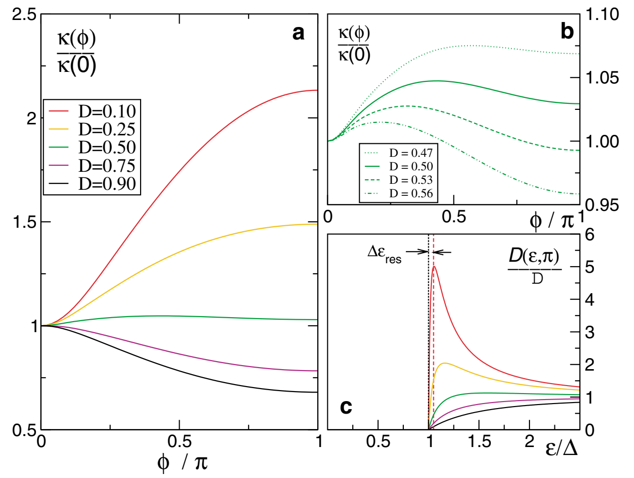

FIG. 2 (color online). (a) The thermal current as a function of ϕ and barrier transparency, D. The thermal conductance is normalized for each D by its value at $\phi = 0$. (b) Nonmonotonic oscillations of $\kappa(\phi)$ for $D \simeq 0.5$. (c) Normalized transmission coefficient at $\phi = \pi$ showing the resonant transmission for $D \ll 1$, and the suppressed transmission for $D \to 1$. The resonance peak is at $\epsilon_{\text{res}} \simeq \Delta(1 + D/2)$.

We compare Eq. (4) for $D \ll 1$ with the heat current obtained by perturbation theory from a tunneling Hamiltonian (TH) description of SIS tunnel junctions. Based on the TH method Guttman *et al.* [5] obtained a heat current of the form $I_0 + I_1\cos\phi$. If we expand $\mathfrak{D}$ to leading order in the barrier transmission probability, D, we obtain the TH result for the conductance from the linear response limit of Eq. (2) of Ref. [5(b)],

$$\kappa^{\text{tH}} = \mathcal{A} N_f v_f D \int_\Delta^\infty d\epsilon\,\epsilon \frac{\epsilon^2 - \Delta^2\cos\phi}{\epsilon^2 - \Delta^2}\left(\frac{\partial f}{\partial T}\right). \quad (7)$$

This result has an unphysical divergence due to the singularity at $\epsilon = \Delta$. In the TH method the divergence is regulated by an *ad hoc* procedure. Guttman *et al.* [5] required a finite temperature difference and $\Delta(T_1) \neq \Delta(T_2)$ for the gaps of the two superconductors. However, Eqs. (4)–(6) for arbitrary transparency show that the unphysical divergence is not a singularity of the limit $\delta T \to 0$, but a failure of perturbation theory in the tunnel Hamiltonian, which does not include the change in the spectrum near the contact. The bound state and the resonance regulate the singularity obtained in perturbation theory and lead to a thermal conductance that is non-analytic, $\sim D\ln D$, but vanishes for $D \to 0$. The result for $\kappa(\phi)$ to order D also has nonperturbative corrections to the phase modulation of the conductance, which includes terms $\propto \cos\phi$ as well as $\sin^2\frac{\phi}{2}\ln(\sin^2\frac{\phi}{2})$,

$$\kappa(\phi) = \kappa_0 - \kappa_1\sin^2\frac{\phi}{2}\ln\left(\sin^2\frac{\phi}{2}\right) + \kappa_2\sin^2\frac{\phi}{2}, \quad (8)$$

where $\kappa_1 = k\,\text{sech}^2(\Delta/2T)$, $k = \mathcal{A}\,N_f v_f D(\Delta^3/4T^2)$,

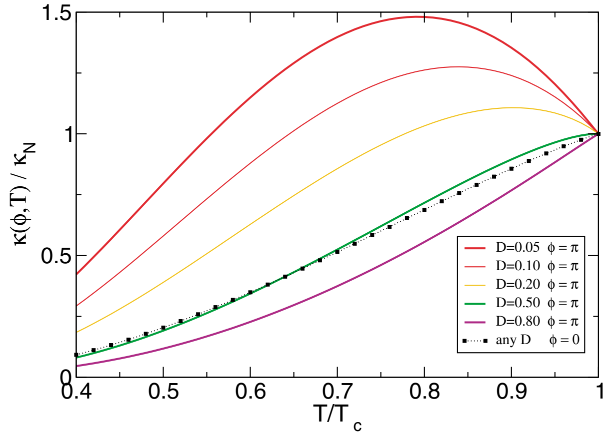

FIG. 3 (color online). The temperature dependence of phase modulation of the thermal conductance for $\phi = \pi$, normalized by the conductance at T_c, $\kappa_N = \frac{\pi^2}{6}\mathcal{A}N_f v_f T_c D$. Shown for comparison is the conductance for $\phi = 0$.

$\kappa_0 = k\int_1^\infty dx x^2 \mathrm{sech}^2(x\Delta/2T)$, $\kappa_2 = -(1+\ln D)\kappa_1 + k\{(4T/\Delta)[1-\tanh(\Delta/2T)] + c\}$, and $c = 2\int_0^\infty dx \ln x[1 + (\Delta/T)\sqrt{x^2+1}\tanh(\Delta\sqrt{x^2+1}/2T)] \times (x^2+1)^{-3/2}\mathrm{sech}^2(\Delta\sqrt{x^2+1}/2T)$. The relative magnitude of the nonperturbative correction, κ_1/κ_2, is approximately 25% for $D = 0.01$ at $T/T_c = 0.8$ and increases with decreasing temperature.

For general transparency the magnitude of the phase modulation of the thermal conductance is a maximum for $\phi = \pi$, except for a small range of barriers with $D \approx 1/2$. The temperature dependence of the conductance for $\phi = \pi$ is plotted in Fig. 3. The thermal conductance for $\phi = 0$ is also shown for comparison. For moderate to high-transmission junctions ($D \gtrsim 1/2$) the thermal conductance for $\phi \neq 0$ is suppressed relative to the conductance at $\phi = 0$ at all temperatures. However, for low-transparency junctions ($D \ll 1$) resonant transmission of quasiparticles just above the gap edge leads to an *increase* in the conductance when T drops below T_c. This effect is pronounced for junctions with $D \lesssim 0.2$; its observation would provide a test of this theory of phase-induced resonant transmission of quasiparticles.

The results above were obtained in the clean limit for the superconducting leads. In the diffusive limit, $k_f^{-1} \ll \ell \ll \xi_0$, where ℓ is the elastic mean-free path, the excitations and pairing correlations are governed by Usadel's diffusion equations [14] for the Fermi-surface averaged propagators, $\hat{\mathfrak{g}}_j^{\mathrm{A,R,K}} = \int d^2\mathbf{p}_f \hat{\mathfrak{G}}_j^{\mathrm{A,R,K}}(\epsilon, \mathbf{p}_f; \mathbf{R})$. For an ScS contact Nazarov derived a boundary condition for the propagator in diffusive conductors [15],

$$\sigma_2 \check{\mathfrak{g}}\partial_z \check{\mathfrak{g}}|_2 = \frac{1}{\mathcal{A}R_b}\left\langle \frac{2D[\check{\mathfrak{g}}_2, \check{\mathfrak{g}}_1]}{4 + D(\{\check{\mathfrak{g}}_2, \check{\mathfrak{g}}_1\} - 2)} \right\rangle_D, \qquad (9)$$

where $\check{\mathfrak{g}}_j$ is the Keldysh matrix representation for the $\hat{\mathfrak{g}}_j^{\mathrm{A,R,K}}$, R_b is the barrier resistance for normal leads, and $\langle \cdots \rangle_D = \int dD\rho(D)\ldots / \int dD\rho(D)D$ is an average over a distribution of channels with transmission coefficient D characterizing the interface. For a single channel contact with transmission coefficient D application of Eq. (9) yields the result from Eqs. (4)–(6) obtained in the ballistic limit with the replacement $N_f v_f \mathcal{A}D/4 \to 1/2e^2R_b$. Thus, the phase modulation of the thermal conductance of small Josephson weak links are the same in the clean and diffusive limits. This result is due to the cancellation of impurity renormalization of the diagonal and off-diagonal self-energies in the propagators for s-wave superconductors up to order a/ξ_Δ, and that Nazarov's boundary condition is based on a junction model with a central layer and interface described by Zaitsev's boundary condition.

In summary, we have presented a theory for heat transport through Josephson weak links. For high-transmission junctions the reduction in states with $\epsilon \geq \Delta$, resulting from the formation of an ABS near the Fermi level, leads to a suppression of the conductance near $\phi = \pi$. For small transparency, the presence of a shallow bound state produces a resonance in the continuum just above the gap edge. This leads to an increase in conductance as the temperature drops below T_c for junctions with $\phi \approx \pi$. For a single channel contact, these results are insensitive to impurity scattering and hold in the clean and dirty limits.

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