Route to Nonlocality and Observation of Accessible Solitons

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(Received 21 August 2002; published 11 August 2003)

We develop a general theory of spatial solitons in a liquid crystalline medium exhibiting a nonlinearity with an arbitrary degree of effective nonlocality. The model accounts the observability of *accessible solitons* and establishes an important link with parametric solitons.

DOI: 10.1103/PhysRevLett.91.073901 PACS numbers: 42.65.Tg, 42.65.Jx, 42.70.Df

In various areas of applied nonlinear science, nonlocality plays a relevant role and radically affects the underlying physics. Some striking evidences are found in plasma physics [1–3], or in Bose-Einstein condensates (BEC) [4,5], where, contrary to the prediction of purely local nonlinear models, nonlocality may give rise to, or prevent, the collapse of a (plasma- or matter-)wave. In nonlinear optics, particularly when dealing with self-localization and solitary waves, nonlocality is often associated to time-domain phenomena through a retarded response (see, e.g., [6,7]); spatially nonlocal effects have been associated to photorefractive [8–11] and thermal or diffusive responses [12,13]. To assess the role of nonlocality, theoretical studies tend to distinguish between *highly* and *weakly* nonlocal behaviors [14–16], by comparing the spatial extent of the material response (the so-called kernel function) and the optical beam waist. Specific kernel functions, however, strongly depend on the physical system and, as in the case of BEC [5], they are hard to determine and apply to experimental results. On the other hand, they are at the basis of the theory of spatial optical solitons (SOS) in highly nonlocal media.

SOS have become the subject of intense theoretical and experimental investigations, both on the grounds of their packet nature and in view of applications, particularly in the exploitation of their wave-guiding character [17]. Diverse material properties have been studied in conjunction with SOS existence and properties, including various mechanisms able to counteract diffraction in one or both transverse dimensions [18,19]. Spatial solitons due to a local nonlinearity have been known since the original work of Chiao, Garmire, and Townes with reference to Kerr media [20]. In 1997, Snyder and Mitchell [14] investigated SOS in a highly nonlocal system, i.e., a medium exhibiting a power — rather than intensity — dependent refractive index. They introduced the term *accessible solitons* for those spatial solitary waves, owing to the simplicity of the theory and transverse profiles obeying the two-dimensional equation of a quantum harmonic oscillator (which gives a Gaussian profile similar to the so-called "gaussons" [21]). Shen pointed to this connection as an intriguing one between distinct fields of modern physics and, provided the required large correlation lengths could be made available, to the demonstration of *accessible solitons* as a challenge well worth undertaking [22].

In this Letter, we introduce a model able to describe optical spatial solitons and the smooth transition from the purely local (in the limit of a Kerr nonlinearity) to the entirely nonlocal case. While pursuing a general theory, however, we chose to address a specific and available nonlocal system, i.e., nematic liquid crystals (NLC) in a planar cell. NLC have been proven to exhibit a substantial nonlocal nonlinearity of molecular origin [23] and to support $(2+1)$-dimensional spatial solitons [24], even in the case of spatially incoherent excitations [25]. After deriving the ruling equations and defining a suitable nonlocal parameter able to span the soliton family from pure-Kerr or Townes-like (T) to highly nonlocal or *accessible* (A) solitons, we will outline the rather remarkable connection between our model and the equations describing quadratic two-color solitons (or *simultons*) in parametric media [26].

Let us consider the simple geometry sketched in Fig. 1: a planar glass cell containing an undoped NLC with a preset orientation of its molecular director. The aligned liquid crystal, anchored at the bounding interfaces, behaves as a positive uniaxial with $n_{\|} = n_{\hat{z}}$ and $n_{\perp} = n_{\hat{x}}$. In the presence of an external quasistatic (electric or magnetic) field or special anchoring at the interfaces, the refractive index $n(\theta)$ in the (x, z) principal plane can

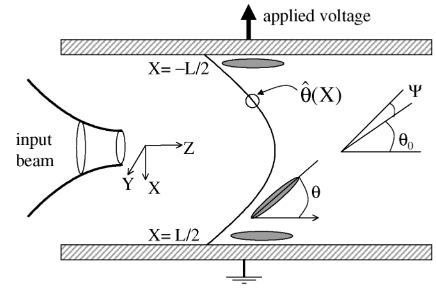

FIG. 1. Planarly aligned nematic liquid crystal cell for the observation of spatial solitons. $\hat{\theta}(X)$ is the low-frequency voltage-induced angle displacement. $\theta_0 = \hat{\theta}(0)$ and Ψ is the perturbation due to the propagating optical field.

exhibit a distribution along x, with $n_\perp < n[\theta(x)] < n_\parallel$ [27]. For a light beam linearly polarized along x and propagating along z, with transverse size well below the cell thickness L, neglecting vectorial effects and adopting the paraxial approximation, the evolution of the optical envelope A is described by the Foch-Leontovich equation:

$$2ik\frac{\partial A}{\partial Z} + \nabla^2_{XY}A + k_0^2 n_a^2[\sin(\theta)^2 - \sin(\theta_0)^2]A = 0, \quad (1)$$

where k_0 is the vacuum wave number, $n_a^2 = n_\parallel^2 - n_\perp^2$ the optical anisotropy, $k^2 = k_0^2[n_\perp^2 + n_a^2 \sin(\theta_0^2)^2]$, θ the tilt angle of the NLC director, and θ_0 the tilt in the absence of a light beam. When an external electric field applied along x and an optical excitation as in Eq. (1) are present, θ is subject to reorientation according to [23]

$$K\frac{\partial^2\theta}{\partial Z^2} + K\nabla^2_{XY}\theta + \frac{\Delta\epsilon_{\mathrm{RF}}E^2}{2}\sin(2\theta) + \frac{\epsilon_0 n_a^2|A|^2}{4}\sin(2\theta) = 0, \quad (2)$$

with K the relevant elastic constant taken equal for splay, bend, and twist, $\Delta\epsilon_{\mathrm{RF}}$ the low-frequency anisotropy, and E the rms value of the quasistatic field.

In the absence of a light wave, therefore, the orientation angle $\hat{\theta}$ is determined exclusively by E and, due to symmetry, depends only on X [23]:

$$K\frac{d^2\hat{\theta}}{dX^2} + \frac{\Delta\epsilon_{\mathrm{RF}}E^2}{2}\sin(2\hat{\theta}) = 0. \quad (3)$$

In our case, the boundary conditions correspond to the planar alignment: $\theta(X = -L/2) = \theta(X = L/2) = 0$. In the general case, the angle distribution can be written as

$$\theta(X, Y, Z) = \hat{\theta}(X) + \frac{\hat{\theta}(X)}{\theta_0}\Psi(X, Y, Z). \quad (4)$$

Taking the cell much larger than the beam waist, we can use (3) and (4) in (2) and neglect the derivative of $\hat{\theta}$. For $\hat{\theta} \cong \theta_0$, at the first order in Ψ we obtain

$$\begin{aligned} &2ik\frac{\partial A}{\partial Z} + \nabla^2_{XY}A + k_0^2 n_a^2 \Psi A = 0, \\ &K\nabla^2_{XYZ}\Psi - \frac{2\Delta\epsilon_{\mathrm{RF}}E^2}{\pi}\Psi + \frac{\epsilon_0 n_a^2}{4}|A|^2 = 0, \end{aligned} \quad (5)$$

having chosen $\theta_0 = \pi/4$ in order to maximize the nonlinear response [27]. We write (5) in a dimensionless form by setting $A = (A_c/\alpha)a(R/R_c\sqrt{\alpha}, Z/\alpha Z_c)\exp(iZ/\alpha Z_c)$, $\Psi = (\Psi_c/\alpha)\psi(R/R_c\sqrt{\alpha}, Z/\alpha Z_c)$ with $A_c^2 = 8\Delta\epsilon_{\mathrm{RF}}^2 E^4/\pi^2\epsilon_0 k_0^2 n_a^4 K$, $Z_c = 2kR_c^2$, $R_c^2 = \pi K/2\Delta\epsilon_{\mathrm{RF}}E^2$, $\Psi_c = 2\Delta\epsilon_{\mathrm{RF}}E^2/\pi k_0^2 n_a^2 K$, and $\epsilon = \Delta\epsilon_{\mathrm{RF}}E^2/2\pi k^2 K$. α is a free parameter, to be used hereafter to trace the family of spatial solitary waves, and $(x, y, z) = (X/\sqrt{\alpha}R_c, Y/\sqrt{\alpha}R_c, Z/\alpha Z_c)$ are normalized coordinates. The resulting system is

$$\begin{aligned} &i\frac{\partial a}{\partial z} + \nabla^2 a - a + a\psi = 0, \\ &\frac{\epsilon}{\alpha}\frac{\partial^2\psi}{\partial z^2} + \nabla^2\psi - \alpha\psi + \frac{1}{2}|a|^2 = 0. \end{aligned} \quad (6)$$

To underline the physical meaning of the parameter α, let us consider the case $\epsilon = 0$ and formally write the solution of the reorientation equation as $\psi = (1 - \nabla^2/\alpha)^{-1}|a|^2/(2\alpha)$. For large α, we obtain the following for the field envelope:

$$i\frac{\partial a}{\partial z} + \nabla^2 a + a\left(1 + \frac{\nabla^2}{\alpha}\right)\frac{|a|^2}{2\alpha} = 0. \quad (7)$$

Equation (7) rules collapse-free weakly nonlocal media [3], while the neglecting of terms such as $O(1/\alpha^2)$ describes light propagation in Kerr media which, on the contrary, are subject to catastrophic self-focusing. Reducing α increases the *degree of nonlocality* in the interaction between the medium and the optical beam. In the following, we will employ α as an arbitrary parameter spanning the whole family of SOS; we expect large α values to be associated to T solitons, whereas small α will address A solitons. Note that, according to [3], when $\epsilon = 0$ the fundamental solitary wave solutions of Eqs. (6) are stable because they realize an absolute minimum of the Hamiltonian.

Solitary solutions of (6) are defined by $\partial_z = 0$. Without loss of generality, taking a real valued we have

$$\nabla^2 a - a + a\psi = 0, \qquad \nabla^2\psi - \alpha\psi + \frac{a^2}{2} = 0. \quad (8)$$

Noteworthy enough, system (8) is identical to what determines the profile of parametric spatial solitary waves in $\chi^{(2)}$ media [19,26]. This makes an unexpected connection between self-trapped beams in two distinct physical systems encompassing rather diverse nonlinear optical responses: the ultrafast electronic nonlinearity of quadratic crystals and the slow molecular reorientation of liquid crystals. The similarity is limited only to the profiles of the solitary waves, while their dynamic properties, such as stability, are quite different. As shown below in a practical case, to a first approximation ϵ is negligible, thus the angle "adiabatically follows" the light beam; in general this is not true for the harmonic field of a parametric soliton. It is well known, in fact, that for small α the ψ field (the second-harmonic for $\chi^{(2)}$ crystals, and the reorientation for NLC) is much wider than the a field (see Torruellas *et al.* in [19]). The opposite holds true for $\alpha \to \infty$: In the $\chi^{(2)}$-SOS literature, this is the *Kerr limit*, its dynamics resembling that of $\chi^{(3)}$ materials. In NLC, for optical beams much wider than the reorientation profile, T solitons approximate well the solution (in the framework of the validity of the large cell approximation). The angle-profile perturbation is localized close to the optical beam axis, as typically pointed out when addressing the Kerr-like response of

NLC [23,27]. Here we are rather interested in the opposite limit.

In Fig. 2, we graph the numerical solutions of Eqs. (6) as obtained by a relaxation procedure. The ratio ρ between SOS beam and angle waists (standard deviation) is plotted versus α. When α approaches zero, i.e., for a response length extending well beyond the optical waist, the ψ field is much wider than the a field. This is the high-nonlocality regime. In the figure we indicate the two opposite limits, namely A and T solitons. In the A limit, however, it is worth proving that Eqs. (6) reduce to the model in [14]. The authors in [14] postulated a highly nonlocal medium in which the index of refraction could be expressed as $n^2 = n_0^2 - \alpha_0^2(\mathcal{P})R^2$, with $\mathcal{P}$ the optical power. Here we show that the NLC reorientational nonlinearity does indeed exhibit such a feature, as speculated rather skeptically in [22].

To study the highly nonlocal regime ($\alpha \to 0$), first we solve Eqs. (8), in the regions around $r = 0$ ($r = \sqrt{x^2 + y^2}$) and $r \to \infty$, and then match the resulting expressions. For $r \cong 0$, we introduce the expanded variables [28] $\zeta = r/\alpha^{1/2}$ and $\xi = r/\alpha^{1/4}$. Next, we express the field as $a = (\hat{a}/\sqrt{\alpha})f(\xi)$ with $f(0) = 1$, while $\psi = \psi^{(0)}(\zeta) + (1/\sqrt{\alpha})\psi^{(-1)}(\zeta)$. At order $O(1/\alpha^{1/2})$, we obtain the following for ψ:

$$\psi^{(0)}_{\zeta\zeta} + (1/\zeta)\psi^{(0)}_{\zeta} + (1/2)\hat{a}^2 f^2(\alpha^{1/4}\zeta) = 0. \tag{9}$$

For $\alpha \to 0$ with ζ fixed, the solution reads $\psi^{(0)} = \psi_0 - \hat{a}^2 r^2/8\alpha = \psi_0 - \hat{a}^2\xi^2/8\sqrt{\alpha}$. At order $O(1/\alpha^{3/2})$, we have $\psi^{(-1)}_{\zeta} = 0$; at order $O(1/\alpha^{1/2})$, the field equation reduces to the harmonic oscillator equation in f:

$$f_{\xi\xi} + (1/\xi)f_{\xi} + \psi^{(-1)}f - (1/8)\hat{a}^2\xi^2 f = 0, \tag{10}$$

corresponding to the result in [14], and $\psi_0 = 1$ at order $O(1)$. After some algebra, the perturbative approach provides the SOS profile near the origin ($\psi^{(-1)} = 2/\xi_0^2$):

$$a = \frac{2\sqrt{2}}{\sqrt{\alpha}\xi_0^2} e^{-[r^2/(2\sqrt{\alpha}\xi_0^2)]}, \qquad \psi = 1 + \frac{2}{\sqrt{\alpha}\xi_0^2} - \frac{r^2}{\alpha\xi_0^4}, \tag{11}$$

with ξ_0 an arbitrary parameter. The appearance of another free parameter reflects that, when $\alpha = 0$, Eqs. (6) are invariant with respect to the transformation $(x, y) \to (x/\mu, y/\mu)$, $a \to \mu a$, $\psi \to 1 - \mu + \mu\psi$, with μ arbitrary. The physical meaning of (11) can be elucidated by expressing them in terms of the normalized beam power $P = \iint a^2\, dx\, dy$: $a = (\sqrt{2}P/4\pi)\exp(-r^2P/16\pi)$, and $\psi = 1 + (P/4\pi) - (P/8\pi)r^2$. The latter results can be also obtained by using a multiple scales expansion in $\sqrt{P}r$, Pr, ... For the field intensity in the original variables ($R = \sqrt{X^2 + Y^2}$), we have

$$I = \frac{|A|^2}{2\eta} = \frac{\mathcal{P}^2}{\pi R_c^2 \mathcal{P}_c} \exp\left(-\frac{R^2}{R_c^2}\frac{\mathcal{P}}{\mathcal{P}_c}\right). \tag{12}$$

In (12), Z_0 and $\eta = Z_0/n$ are vacuum and medium impedances, respectively, c the speed of light in vacuum, $n = k/k_0 = \sqrt{n_\perp + n_a^2/2}$ when $\theta_0 = \pi/4$, and $\mathcal{P}_c$ a reference power which depends on material properties and cell polarization: $\mathcal{P}_c = 16cn\Delta\epsilon_{\mathrm{RF}}E^2/k_0^2 n_a^4$. The soliton profile is Gaussian and completely determined by $\mathcal{P}$. Rendering explicit the relation between $\mathcal{P}$ and the (intensity) waist $\mathcal{W}$, its existence curve is

$$\frac{\mathcal{P}}{\mathcal{P}_c} = \frac{R_c^2}{\mathcal{W}^2}. \tag{13}$$

Let us now investigate the region of large r. According to Eqs. (6), the asymptotic behavior of ψ as $r \to \infty$ is given by the modified Bessel function: $\psi \to GK_0(\sqrt{\alpha}r)$, governing the angle decay far from the beam axis (G is a constant to be determined). The angle in (11) and its derivative must match the expression at infinity, providing the *turning point* r_T between the two regions: in the presence ($r \cong 0$) and in the absence ($r \to \infty$) of the optical excitation, respectively. We end up with

$$1 + \frac{P}{4\pi} - r_T^2\left(\frac{P}{8\pi}\right)^2 = \left(\frac{P}{8\pi}\right)^2 \frac{2r_T K_0(\sqrt{\alpha}r)}{\sqrt{\alpha}K_1(\sqrt{\alpha}r)}, \tag{14}$$

which can be further simplified by taking into account that, for large arguments, the Bessel functions K_0 and K_1 have the same asymptotic behavior. The approximated solution is $r_T = \sqrt{\alpha}(8\pi/P)$; being $\mathcal{R}_T = \sqrt{\alpha}R_c r_T$ it reads $\mathcal{R}_T/\mathcal{R}_c = \mathcal{P}_c/\mathcal{P}$. When $\mathcal{P} \gg \mathcal{P}_c$, the profile of the angle is dominated by the modified Bessel function, and decays with typical length R_c. Therefore, it seems natural to label "highly nonlocal" a regime when the angle profile is much larger than the beam waist, such that $\mathcal{R}_c \gg \mathcal{W}$. When the power is much greater than $\mathcal{P}_c$ or, equivalently, when the soliton waist is much smaller that the extent $\mathcal{R}_c$ of the elastic response, we are in the

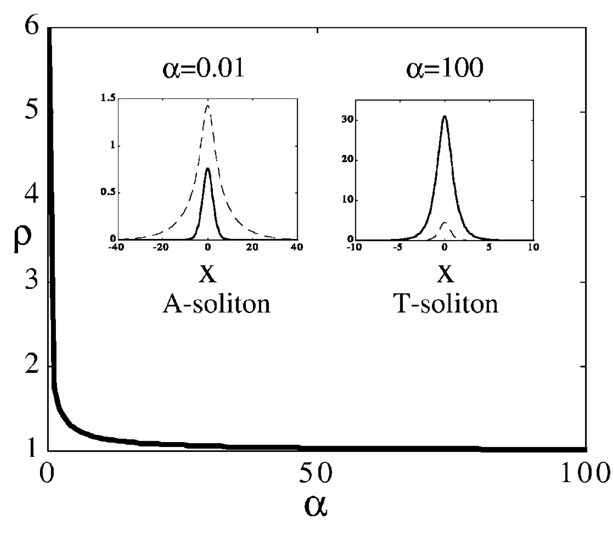

FIG. 2. Ratio ρ between the waists of the angle ψ and the field a from Eq. (8). The insets show two different profiles (dashed line: ψ; solid line: a) for values of α addressing the two limits in the family of solitary waves.

A-soliton regime. The power-dependent perturbation of the refractive index is

$$\Delta n(R, \mathcal{P}) = n_c\left(\frac{2\mathcal{P}}{\mathcal{P}_c} - \frac{R^2}{R_c^2}\frac{\mathcal{P}^2}{\mathcal{P}_c^2}\right), \qquad (15)$$

with $n_c = \Delta\epsilon_{\rm RF}E^2/\pi n k_0^2 K$. Moreover, our approach enables one to go beyond the harmonic oscillator. By introducing the new transverse scale $\sigma = r/\alpha^{3/8}$, we can solve the resulting equation for $\psi^{(0)}$ at order $O(\alpha^{1/4})$: $\psi^{(0)}_{\sigma\sigma} + (1/\sigma)\psi^{(0)} - 4\sigma^2/\xi_0^6 = 0$. After some algebra, we obtain the new approximation in normalized units: $\psi = 1 + (P/4\pi) - (P/8\pi)r^2 + (P/8\pi)^3 r^4/4$. Then, the corresponding power-dependent refractive index perturbation reads

$$\Delta n(R, \mathcal{P}) = n_c\left[\frac{2\mathcal{P}}{\mathcal{P}_c} - \frac{R^2}{R_c^2}\left(\frac{\mathcal{P}}{\mathcal{P}_c}\right)^2 + \frac{R^4}{4R_c^4}\left(\frac{\mathcal{P}}{\mathcal{P}_c}\right)^3\right]. \qquad (16)$$

Thus, higher-order approximations imply anharmonicity of the nonlocal potential, with a refractive index still depending on power. As the latter increases, higher powers of the ratio $\mathcal{P}/\mathcal{P}_c$ must be taken into account, similar to local media with regards to the powers of the intensity.

Finally, comparing our theory with an actual experimental geometry, such as employed in [24], typical parameters for a 514 nm wavelength and the E7 NLC (in SI units) are $n_a = 1$, $K = 10^{-11}$, $E = 1.3 \times 10^{-4}$, $L = 75 \times 10^{-6}$, $\Delta\epsilon_{\rm RF} = 20\,\epsilon_0$. Correspondingly, $\mathcal{P}_c = 2 \times 10^{-6}$ W, $R_c = 22$ μm, and $\epsilon = 10^{-6}$. Since $\mathcal{P}$ inside the cell was of the order of 0.1 mW, we may state that in experiments the highly nonlocal regime ($\mathcal{P} \gg \mathcal{P}_c$) is being addressed. The soliton waist, after (13), is of the order of 3 μm and in agreement with the reported results. Note that $\Delta n(0, \mathcal{P}) \cong 5 \times 10^{-4}$, and the angle perturbation Ψ of the order of 10^{-3} rad, thus justifying the adopted model.

In conclusion, for the first time to our knowledge, we have presented a self-consistent analytical theory of two-dimensional spatial solitary waves in nonlocal media. Our model has an intriguing unifying character, as it embraces several physical systems in which light self-trapping has recently been investigated. We believe that most of the observed SOS in nematic liquid crystals are indeed *accessible solitons*, inasmuch as NLC are highly nonlocal. This shines new light on self-localization in liquid crystals. Furthermore, we have presented the first derivation of a power-dependent constitutive relation for a real physical system, never reported elsewhere. We confide that our results will stimulate new experiments towards a deeper understanding of self-trapping in (highly) nonlocal nonlinear media and the development of novel all-optical devices [29].

C. C. acknowledges the Tronchetti Provera Foundation for a generous grant. This work was supported by INFM (Advanced Research Project "SPASONELIC").

*URL: http://optow.ele.uniroma3.it/opto_2002.shtml

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