Entangling Strings of Neutral Atoms in 1D Atomic Pipeline Structures

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We study a string of neutral atoms with nearest neighbor interaction in a 1D beam splitter configuration, where the longitudinal motion is controlled by a moving optical lattice potential. The dynamics of the atoms crossing the beam splitter maps to a 1D spin model with controllable time dependent parameters, which allows the creation of maximally entangled states of atoms by crossing a quantum phase transition. Furthermore, we show that this system realizes protected quantum memory, and we discuss the implementation of one- and two-qubit gates in this setup.

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The recent development of optical and magnetic microtraps allows the confinement of cold atoms in effective 1D "pipeline" geometries, where the transverse quantum motion is frozen out [1]. Variants of these trap designs promise the realization of beam splitters, and thus atomic interferometry "on a chip." Usually one envisions that atoms are injected one by one into these pipelines, where the source of cold atoms is provided by a Bose-Einstein condensate. Instead we will study below collective beam splitter setups which allow the *generation of entangled strings of atoms* in 1D trapping configurations with applications in interferometry and quantum computing.

To this end, we assume that the longitudinal motion of the atoms is controlled by storing atoms in a 1D optical lattice potential generated by a standing light laser field. In the transverse direction, the particles are confined by a double well potential [see Fig. 1(a)] where we assume that the optical lattice stores exactly one atom per lattice site (i.e., one atom per double well). The preparation of such a Mott insulating state has been reported in a recent experiment, by loading of atoms from a Bose-Einstein condensate via a superfluid-Mott insulator quantum phase transition (QPT) [2]. This setup by itself is an interesting extension of the standard "interferometry on a chip," as it eliminates collisional shifts since atoms stored on different lattice sites never collide. Furthermore, the atoms are supposed to be initially in the ground state which is a spatial superposition of the particles in the two transverse wells [region (I) of Fig. 1(a)]. By moving the lattice, we can drag the atomic chain "by hand" across the beam splitter while we increase the distance between the transverse wells adiabatically depending on the position of the atoms [i.e., we decrease the tunneling J^x between the wells, see region (II) of Fig. 1(a)]. On the other hand, the use of optical lattices allows the engineering of coherent interactions between adjacent atoms [nearest neighbor interaction W in Fig. 1(a)]. This can be obtained either by cold collisions and moving optical lattices [3–5], the remarkably strong dipole-dipole couplings of laser excited Rydberg atoms [6], or by dipole-dipole coupling of cold heteronuclear molecules [7]. Together with appropriate detection methods such as fluorescence imaging, these controllable interactions provide us with the tools to generate entanglement of the 1D chain of atoms.

We will study the dynamics of the beam splitter setup indicated in Fig. 1. In particular, we will investigate (i) how to generate a maximally entangled state of atoms. (ii) We will establish the formal equivalence of our model with well-studied models of spin chains. We will show that the system dynamics is a physical realization of a textbook model of a QPT with completely controllable (time dependent) parameters [8]. Thus, our setup provides an example of engineering a maximally entangled state from a product state via a QPT. (iii) Finally, the present setup implements the spin analogue [9] of Kitaev's protected quantum memory [10], where qubits are represented by Majorana fermions, which provide a stable way to store quantum information due to an excitation gap [11]. Our setup allows one to perform single and (collectively enhanced) two-qubit operations.

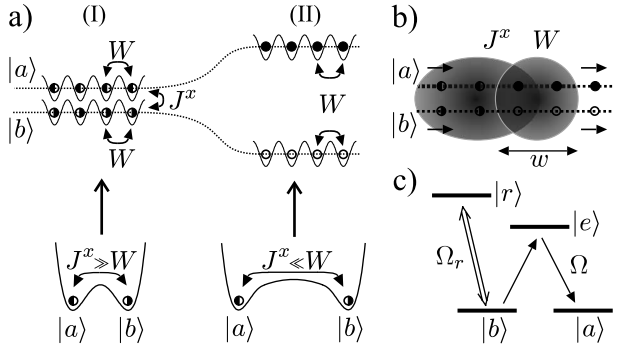

FIG. 1. *External beam splitter*: (a) Atoms before (I) and after (II) the separation. The nearest neighbor interaction is denoted by W and J^x is the hopping matrix element between the two states $|a\rangle$ and $|b\rangle$ of the transverse trapping potential. *Internal beam splitter*: (b) Atoms in two different internal states $|a\rangle$ and $|b\rangle$ enter the beam splitter. These internal states are coupled by a Raman transition [see (c)] with a Rabi frequency $J^x = \Omega$. A laser excited Rydberg state $|r\rangle$ realizes the off site interaction W with w the width of the interaction zone.

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We consider a 1D chain of N atoms with modes $|a\rangle$ and $|b\rangle$ stored in an optical lattice with a lattice constant $\lambda/2$ determined by the wavelength λ of the laser. The modes correspond either to two spatial modes in a double well structure, where the tunneling provides a coupling [external beam splitter in Fig. 1(a)], or to two internal atomic states connected via a Raman process [cf. Figs. 1(b) and 1(c)]. We suppress hopping of the atoms between adjacent lattice sites by a sufficiently large potential barrier. This leads to an on site interaction $U \to \infty$, and we assume to have commensurate filling of one particle per lattice site. Following [3,12], we derive a Hubbard Hamiltonian

$$H(t) = 2\sum_{l=1}^{N-1} W_l(t)(a_{l+1}^\dagger a_{l+1} a_l^\dagger a_l + b_{l+1}^\dagger b_{l+1} b_l^\dagger b_l) - \sum_{l=1}^{N}[J_l^x(t)(a_l^\dagger b_l + a_l b_l^\dagger) + J_l^z(t)(a_l^\dagger a_l - b_l^\dagger b_l)]. \quad (1)$$

Here J_l^x describes coupling between $|a\rangle$ and $|b\rangle$ while the operators a_l, b_l are bosonic annihilation operators for particles in these modes at site l with $[a_l, b_j] = [a_l, b_j^\dagger] = 0$. A term J_l^z emerges from an additional state dependent superimposed trapping potential. We introduce the spin notation $\sigma_l^x = a_l^\dagger b_l + a_l b_l^\dagger$, $\sigma_l^z = a_l^\dagger a_l - b_l^\dagger b_l$, and $\sigma_l^y = i(a_l b_l^\dagger - a_l^\dagger b_l)$ which for $n_l = a_l^\dagger a_l + b_l^\dagger b_l \equiv 1$ are Pauli operators and rewrite the Hamiltonian (1) as $H_S(t) = \sum_{l=1}^{N-1} W_l(t)\sigma_l^z\sigma_{l+1}^z - \sum_{l=1}^{N} \mathbf{J}_l(t)\cdot\boldsymbol{\sigma}_l$. Thus, our setup is formally equivalent to an Ising chain of N spins in a magnetic field $\mathbf{J}_l = (J_l^x, J_l^y, J_l^z)$ [13,14].

Entanglement via QPT.—Moving a string of atoms from left to right in the setup of Fig. 1(a), or switching the lasers in Fig. 1(b) and 1(c) amounts to a time dependent change of the parameters from the large tunneling limit $J_l^x(t=0) \gg |W_l|$ to small tunneling $J_l^x(t=T) \to 0$. In the following, we assume that $J_l^{y,z} = 0$ except it is stated differently. In the homogeneous case (i.e., $J_l^x = J^x$, $W_l = W$) the variation of J^x amounts to crossing the critical point at $J^x = W$ of a quantum phase transition [8]. We assume that the atoms are initially prepared in the product state $|\uparrow\uparrow\cdots\uparrow\rangle^x$, where $|\uparrow\rangle_l^x \sim |a\rangle_l + |b\rangle_l$ is a superposition state of the two modes, and which is for $W = 0$ the (paramagnetic) ground state of H_S. Under adiabatic variation of parameters, the system will remain in the ground state and evolve according to ($W < 0$)

$$|+\rangle \equiv |\uparrow\uparrow\cdots\uparrow\rangle^x \to (|\uparrow\uparrow\cdots\uparrow\rangle^z + |\downarrow\downarrow\cdots\downarrow\rangle^z)/\sqrt{2} \equiv (|0\rangle + |1\rangle)/\sqrt{2}, \quad (2)$$

where the states $|\uparrow\rangle_l^z = |a\rangle_l$ and $|\downarrow\rangle_l^z = |b\rangle_l$ correspond to the atoms being in the upper or lower branch of the beam splitter of Fig. 1. The states $|0\rangle$ and $|1\rangle$ are the two degenerate (ferromagnetic) ground states of H_S for $J^x = 0$ with all atoms in either one or the other arm of the beam splitter (see Fig. 1). Thus, the initial *product state* is transformed to a *maximally entangled state* via a quantum phase transition. The intuitive physical picture behind (2) is as follows. Consider atoms moving across the beam splitter one by one. The first atom of the string will end up in the state $|\uparrow\rangle^z + |\downarrow\rangle^z$, and attract the second atom. This leads to a state of the form $|\uparrow\uparrow\rangle^z + |\downarrow\downarrow\rangle^z$. After the last atom has left the interaction zone, the maximally entangled state $|0\rangle + |1\rangle$ has been created.

In the following, we discuss the validity of the adiabatic approximation [Eq. (2)] and thus the usefulness of this scheme by studying the scaling of the fidelity $F = |\langle\psi_{\rm id}|\psi(T)\rangle|^2$ as a function of the length of the string N and the time variation of $J_l^x(t)$ and $W_l(t)$. Here F compares the state $|\psi(T)\rangle$ obtained from a time dependent integration of the Schrödinger equation with the ideal state $|\psi_{\rm id}\rangle \sim |0\rangle + |1\rangle$. This will be done first numerically, followed by analytical calculations and estimates.

Before entering the time dependent case, we note that for the time independent case the Hamiltonian H has been studied extensively [8,15]. For $J_l^z = 0$, it can be fermionized and one obtains $H_F = \sum_\nu \varepsilon_\nu (f_\nu^\dagger f_\nu - 1/2)$ with the elementary excitation energies ε_ν and fermionic annihilation (creation) operators f_ν ($f_\nu^\dagger$). The spectrum for the homogeneous case is shown in Fig. 2(a). For large N, the spectrum of the elementary excitations is characterized by a gap $\Delta = 2|W - J^x|$ for the energetically low lying quasiparticles with the exception (arising

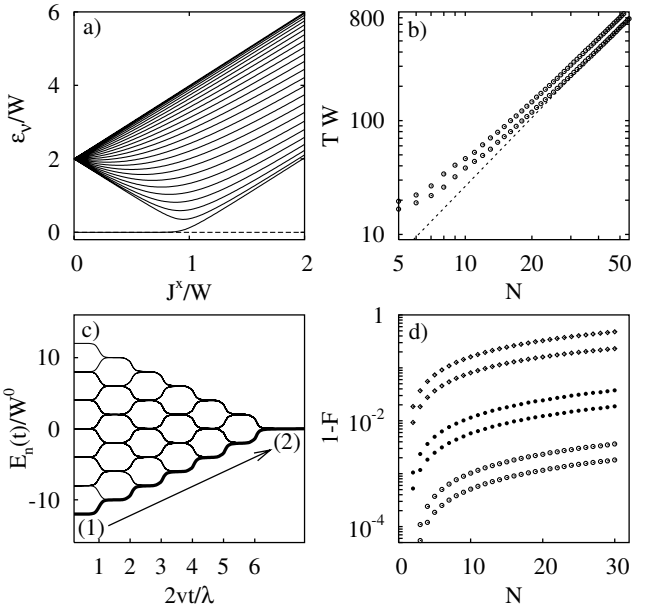

FIG. 2. *Homogeneous setup* $J_l^x \equiv J^x$, $W_l \equiv W$: (a) Elementary excitations ε_ν for $N = 25$ against J^x/W. The dashed horizontal line indicates the ground state energy. (b) Upper bound and lower bound for the time T yielding a fidelity of $F = 95\%$ as a function of N. The dashed line illustrates the N^2 scaling predicted analytically. *Inhomogeneous setup* J_l^x, W_l: (c) Instantaneous eigenenergies E_n for $N = 6$. (d) Upper and lower bounds for the infidelity $1 - F$ against N for constant sweeping speed $v = 0.01\lambda W$ and different interaction zone widths $w = 0.1\lambda$ (diamonds), $w = 0.2\lambda$ (solid circles), and $w = 0.4\lambda$ (open circles) and similar for J^x.

from the free end boundary conditions) that the first excited state becomes degenerate with the ground or vacuum state (here, we do not take into account the second term in H_F; i.e., the vacuum state has zero energy) for $|W| \gg J^x$ [cf. Figure 2(a)]. For $J^x = 0$, the two cat-type ground states $|0\rangle + |1\rangle$ and $|0\rangle - |1\rangle$ correspond to the vacuum and the first excited state of the fermionized system, respectively.

In Fig. 2(b), we plot the numerically calculated operation time T required to perform (2) with a fidelity of $F = 95\%$ for linearly changing the homogeneous couplings $J^x(t)$ against N ($W = \text{const}$). For $N > 20$, we find a (polynomial) scaling of $WT \sim N^2$ for a given infidelity $1 - F$ [cf. Fig. 2(b)] in agreement with the analytical results below. By optimizing the time dependence of $J^x(t)$ we can speed up the entanglement process significantly.

A discussion of the spatially inhomogeneous situation where J_l^x and W_l vary as a function of l corresponding closer to the setup of Fig. 1 is given in Figs. 2(c) and 2(d). For increasing time, the string is moved across a zone of nonvanishing W_l with a maximum W^0 and a width w. Simultaneously, $J_l^x(t)$ is decreased from the initial value to $J_l^x(T) \approx 0$ over a comparable "width" as w for all sites. The corresponding instantaneous time dependent energy levels are shown in Fig. 2(c). Following the lowest energy curve in this diagram adiabatically from (1) to (2) corresponds to $|+\rangle \rightarrow |0\rangle + |1\rangle$. Figure 2(d) shows the infidelity $1 - F$ for finite sweeping speed v against N for different widths w of the interaction zone. The infidelity $1 - F$ decreases rapidly with increasing w and F scales exponentially with N for $w \ll \lambda N$. For $w \gtrsim N\lambda/2$, the above scaling $WT \sim N^2$ is restored.

The numerical calculations behind Fig. 2 were based on a *time independent* Jordan-Wigner transformation of H_S, yielding a quadratic Hamiltonian in fermionic destruction (creation) operators γ_ν ($\gamma_\nu^\dagger$). By introducing Majorana operators $c_{2\nu-1} = (\gamma_\nu + \gamma_\nu^\dagger)/2$, $c_{2\nu} = (\gamma_\nu - \gamma_\nu^\dagger)/(2i)$ [10], we obtain $H_t = i\mathbf{c}^T\mathbf{A}(\mathbf{t})\mathbf{c}$, where the components of $\mathbf{c}$ are the Majorana operators and $\mathbf{A}(\mathbf{t})$ is a $2N \times 2N$ real antisymmetric tridiagonal matrix. The *linear* Heisenberg equations of motion, $\dot{\mathbf{c}} = \mathbf{A}(\mathbf{t})\mathbf{c}$, are then solved numerically. We note that diagonalizing $\mathbf{A}$ in the time independent case yields H_F. For the fidelity F, we use an approximate expression which can be derived as follows: The state $|0\rangle + |1\rangle$ is the vacuum state of the fermionized system at $t = T$. The completeness relation yields $F(T) = 1 - \sum_{\mathbf{n}\neq\mathbf{0}} |\langle\mathbf{n}|\psi(T)\rangle|^2$. Here $|\mathbf{n}\rangle = |n_1, \ldots, n_N\rangle$ with $n_\nu = 0, 1$ the occupation numbers of the instantaneous eigenstates of H_T corresponding to an energy ϵ_ν. The sum in this expression can be reordered, and we obtain $F(T) = 1 - \sum_{m=1}^N P(m)$, where $P(m) = \sum_{\mathbf{n}} |\langle\mathbf{n}|\psi(T)\rangle|^2 \delta_{m,\sum n_i}$ is the probability of having m elementary excitations in the system at time T. By solving the above equation for $\mathbf{c}$, we can in principle calculate the quantities $A_l \equiv \langle(\sum_{m=1}^N f_m^\dagger f_m)^l\rangle = \sum_{m=1}^N P(m) m^l$. The fidelity F is then given by the solution of a system of N linear equations. An approximate fidelity F_l can be obtained by neglecting the probabilities $P(k)$ with $k > l$. We restrict ourselves to $l = 1, 2$ and find $F_1 = 1 - A_1$ and $F_2 = 1 - (3A_1 - A_2)/2$. The exact fidelity is bounded by these quantities: $F_1 \leq F \leq F_2$. Compared to a calculation in the spin picture which requires the solution of $\sim 2^N$ equations, the calculation of A_1 and A_2 can be done by solving a system of $\sim N^2$ differential equations.

Let us turn to the more technical point of analytically estimating the scaling of the fidelity F when the phase transition point is crossed by linearly changing $J^x = \Theta t + W$ with $\Theta = \text{const}$. First, we note that there are no transitions between the ground and the first excited state since they have opposite parity. Close to the phase transition point the energy gap to the remaining excitations $\Delta \approx 0$, and therefore at the time $t = -t_*$ the evolution of the system ceases to be adiabatic and excitations start to be populated. The adiabaticity is restored again at the time $t \sim t_*$, when the gap Δ becomes sufficiently large to prevent further excitations. Then, the relaxation of the new phase occurs separately within different domains, whose sizes are given by the value $l_0(t_*)$ of the correlation length at the time t_*. Close to the phase transition $l_0 \sim \Delta^{-1/2}$ and therefore the domain size scales similar to $l_0(t_*) \sim \Theta^{-1/2}$. The quench through the phase transition point can only be adiabatic if the characteristic size of the domain formed exceeds the size of the system $L \sim N$ and therefore $l_0(t_*) \gtrsim L$, which gives the scaling condition $\Theta \lesssim W^2/N^2$, or $WT \sim N^2$.

Quantum computing model with protected quantum memory.—In the case $W < 0$, the ferromagnetic superposition state is very sensitive to homogeneous distortions of the form $J_l^z = J^z$ which induce a relative phase shift $\exp[i2N \int_0^\tau dt J^z(t)]$ scaling with N [16] between the two states $|0\rangle$ and $|1\rangle$ after a time τ. Therefore, in the external beam splitter setup where these two states are spatially separated, they can be viewed as two arms of a Heisenberg limited interferometer collectively enhanced by a factor N. On the other hand, in the antiferromagnetic case, i.e., for a repulsive interaction $W > 0$, the two degenerate ground states at $J^x = 0$,

$$|0\rangle = |\downarrow\uparrow\cdots\downarrow\uparrow\rangle^z, \qquad |1\rangle = |\uparrow\downarrow\cdots\uparrow\downarrow\rangle^z, \tag{3}$$

are closely related to unpaired Majorana fermions which have been considered as candidates for storing quantum information [9,10]. These states are expected to be insensitive against perturbations since they are separated by a gap of order W from the other states of the system and are connected only via Nth order perturbation theory for *homogeneous* couplings $\mathbf{J}_l = \mathbf{J}$. This yields stability against spin flip errors exponentially increasing with the number of particles in the chain N and is also reflected by the scaling of the energy of the first excited state $\epsilon_1 \sim (J^x/W)^N$ for $J^x < W$ [15]. Furthermore, if we assume that N is even, the states $|0\rangle$ and $|1\rangle$ are completely insensitive to global fluctuations of $\mathbf{J}$ since $\sum_l \sigma_l^z |0\rangle = \sum_l \sigma_l^z |1\rangle = 0$. Then the two states $|0\rangle$ and $|1\rangle$ constitute a decoherence free subspace [17,18] for homogeneous perturbations and

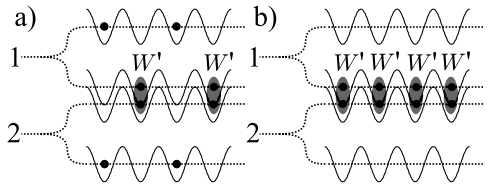

FIG. 3. Collectively enhanced interactions between two strings of atoms 1 and 2. (a) Antiferromagnetic setup: $N/2$ particles of each chain interact with strength W' only if they are in states $|0\rangle_1|1\rangle_2$ or $|1\rangle_1|0\rangle_2$ yielding a phase gate between the two qubits implemented by those chains. (b) Ferromagnetic setup: Entanglement creation between two chains of atoms via interactions W' in the state $|1\rangle_1|0\rangle_2$. Implementations with optical lattices or atom chips, for instance, offer the scalability of the scheme.

can thus be used as qubits which store quantum information reliably.

We will now discuss how to implement single and (collectively enhanced) two-qubit gates and show that our model realizes a quantum computer with protected memory. The idea behind the two-qubit phase gate is summarized in Fig. 3. Selectively overlapping the wave functions of different two-qubit states for a time τ_2 collisional interactions of strength W' between the atoms lead to an entanglement phase $\phi_2 = NW'\tau_2/2$ [3] corresponding to a phase gate with a truth table $|\epsilon_1\rangle|\epsilon_2\rangle \rightarrow \exp\{i\phi_2[(\epsilon_1 + \epsilon_2) \bmod 2]\}|\epsilon_1\rangle|\epsilon_2\rangle$, with $\epsilon_{1,2} = \{0, 1\}$. Single qubit gates correspond to a general unitary rotation of $|0\rangle$ and $|1\rangle$ [Eq. (3)] which can be decomposed in Hadamard gates $|0\rangle, |1\rangle \rightarrow |0\rangle \pm |1\rangle$ and the creation of a relative phase $|\epsilon_1\rangle \rightarrow \exp(i\epsilon_1\phi_1)|\epsilon_1\rangle$. The phase ϕ_1 can be implemented by turning on a trap potential creating a *staggered* offset of the form $J_l^z = J^z(-1)^l$ for a time $\tau_1 = \phi_1/2NJ^z$. The idea behind the Hadamard gate is as follows: At $J^x = J^z = 0$, the states $|0\rangle, |1\rangle$ represent a degenerate eigenspace of H_S. Turning on the field J^x up to $J^x > W$, thus unprotecting the qubit and switching it off when $J^z \neq 0$, will under appropriate conditions induce a rotation in this space. A specific example is illustrated in Fig. 4: (1) at $J^z = 0$ we adiabatically switch on J^x until $J^x > W$, then (2) we increase J^z, (3) we return adiabatically to $J^x = 0$, and, finally, (4) switch off J^z.

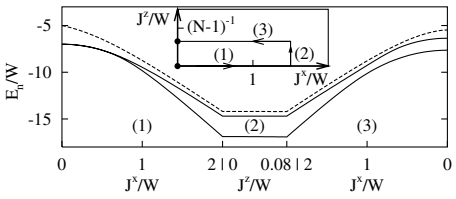

FIG. 4. Illustration of the Hadamard gate for $N = 8$ by adiabatically changing J^z and J^x (unprotecting the quantum memory). We follow the lowest two eigenstates (with energies given by the solid curves) transforming as $|0\rangle + |1\rangle \rightarrow |0\rangle$ and $|0\rangle - |1\rangle \rightarrow |1\rangle$ (up to a dynamical phase) when changing J^x, J^z in three steps (1), (2), (3) [followed by turning off J^z in step (4)] as described in the text. Note that if the condition $J^z < W/(N-1)$ is not fulfilled we get unwanted crossings and the first excited state after step (3) will not be of the form $|1\rangle$. The dashed curve shows the third eigenenergy of the system and the inset the path in the $J^x - J^z$ plane.

We have shown how to generate maximally entangled states of strings of atoms in 1D pipeline configurations. An extension of this setup allows implementations of a quantum computing model with protected qubits.

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