Environment Induced Entanglement in Markovian Dissipative Dynamics

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We show that two, noninteracting two-level systems, immersed in a common bath, can become mutually entangled when evolving according to a Markovian, completely positive reduced dynamics.

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The role of quantum entanglement is of primary importance in quantum information and computation theory. In recent years, a lot of research has been devoted to studying how to entangle two systems by means of a direct interaction between them (see, for instance, [1–5]). In such a context, the presence of an environment, e.g., a generic noisy reservoir or a heat bath, is commonly thought as counteracting entanglement creation, because of its decohering and mixing-enhancing effects.

However, a heat bath can also provide an indirect interaction between otherwise totally decoupled subsystems and thus a means to entangle them. Indeed, this has been explicitly shown in a simple, exactly solvable model [6]. There, correlations between two subsystems are established during a transient phase where the reduced dynamics of the subsystems contains memory effects.

Instead, in this Letter, we study the possibility that entanglement be created by the bath during the Markovian regime through a purely noisy mechanism. We consider two, noninteracting two-level systems, weakly coupled to a common heat bath. We then start with a total Hamiltonian of the form

$$H_{\rm tot} = H_0^{(1)} + H_0^{(2)} + H_B + H_{\rm int}, \tag{1}$$

where $H_0^{(1)}$, $H_0^{(2)}$, and H_B drive the dynamics of the two subsystems and the bath in absence of each other; the interaction term couples each subsystem independently with the bath, and can be taken of the form

$$H_{\rm int} = \sum_{\alpha=1}^{3} (\sigma_\alpha \otimes \mathbf{1}) \otimes V_\alpha + \sum_{\alpha=4}^{6} (\mathbf{1} \otimes \sigma_{\alpha-3}) \otimes V_\alpha, \tag{2}$$

where σ_1, σ_2, and σ_3 are the Pauli matrices. Notice that we allow the subsystems to interact with the bath through different operators V_α, while any direct coupling among themselves has been excluded.

In the weak-coupling limit [7–12], the reduced dynamics of the two two-level systems takes on a Markovian form. Assuming a factorized initial state $\rho \otimes \rho_B$, where ρ is a state of the two subsystems and ρ_B is an equilibrium state of the bath, ρ evolves in time according to a quantum dynamical semigroup of completely positive maps with generator of the Kossakowski-Lindblad form:

$$\partial_t \rho(t) = -i[H, \rho(t)] + L[\rho(t)]. \tag{3}$$

The unitary term is the commutator with an effective Hamiltonian, $H = H^{(1)} + H^{(2)} + H^{(12)}$, consisting of single system pieces, including bath-induced Lamb shifts,

$$H^{(1)} = \sum_{i=1}^{3} H_i^{(1)} (\sigma_i \otimes \mathbf{1}), \qquad H^{(2)} = \sum_{i=1}^{3} H_i^{(2)} (\mathbf{1} \otimes \sigma_i), \tag{4}$$

plus, possibly, a bath generated two-system coupling term

$$H^{(12)} = \sum_{i,j=1}^{3} H_{ij}^{(12)} (\sigma_i \otimes \sigma_j). \tag{5}$$

The dissipative contribution $L[\rho(t)]$ is as follows:

$$L[\rho] = \sum_{\alpha,\beta=1}^{6} \mathcal{D}_{\alpha\beta} \left[\mathcal{F}_\alpha \rho \mathcal{F}_\beta - \frac{1}{2}\{\mathcal{F}_\beta \mathcal{F}_\alpha, \rho\} \right], \tag{6}$$

with $\mathcal{F}_\alpha = \sigma_\alpha \otimes 1$ for $\alpha = 1, 2, 3$, $\mathcal{F}_\alpha = 1 \otimes \sigma_{\alpha-3}$ for $\alpha = 4, 5, 6$, and $\mathcal{D} = \mathcal{D}^\dagger$ a positive 6×6 matrix which guarantees the complete positivity of the evolution. Writing

$$\mathcal{D} = \begin{pmatrix} A & B \\ B^\dagger & C \end{pmatrix} \tag{7}$$

with 3×3 matrices $A = A^\dagger$, $C = C^\dagger$, and B, $L[\rho]$ assumes a form more amenable to a physical interpretation:

$$L[\rho] = \sum_{i,j=1}^{3} \Bigg(A_{ij}\left[(\sigma_i \otimes \mathbf{1}) \rho (\sigma_j \otimes \mathbf{1}) - \frac{1}{2}\{(\sigma_j\sigma_i \otimes \mathbf{1}), \rho\}\right] + C_{ij}\left[(\mathbf{1} \otimes \sigma_i) \rho (\mathbf{1} \otimes \sigma_j) - \frac{1}{2}\{(\mathbf{1} \otimes \sigma_j\sigma_i), \rho\}\right]$$
$$+ B_{ij}\left[(\sigma_i \otimes \mathbf{1}) \rho (\mathbf{1} \otimes \sigma_j) - \frac{1}{2}\{(\sigma_i \otimes \sigma_j), \rho\}\right] + B_{ij}^*\left[(\mathbf{1} \otimes \sigma_j) \rho (\sigma_i \otimes \mathbf{1}) - \frac{1}{2}\{(\sigma_i \otimes \sigma_j), \rho\}\right]\Bigg). \tag{8}$$

A generator of this form has been applied in quantum optics to describe the phenomenon of collective resonance fluorescence (e.g., see [13]). In the above expression, the first two contributions are dissipative terms that affect the first, respectively, the second, system in the absence of the other. On the contrary, the last two pieces represent the way in which the noise may correlate the two subsystems; this effect is present only if matrix B is different from zero.

Remark 1: From the rigorous derivation of Markovian master equations [7,8], one knows that the Hamiltonian terms (4) and (5) and the entries of the matrix $\mathcal{D}$ in (6) contain integrals of two point time-correlation functions of bath operators: $\mathrm{Tr}[\rho_B V_\alpha V_\beta(t)]$. In particular, the matrices $[H_{ij}^{(12)}]$ in (5) and $[B_{ij}]$ in (8) do not vanish only if the bath state ρ_B correlate bath operators V_α pertaining to different subsystems, that is, if the expectations $\mathrm{Tr}[\rho_B V_\alpha V_\beta(t)]$ are nonzero when $1 \le \alpha \le 3$ and $4 \le \beta \le 6$. Only in this case, entanglement has a chance to be created by the action of the bath. Indeed, if $H^{(12)} = 0$ and $B = 0$, the two subsystems evolve independently and initially separable states may become more mixed, but certainly not entangled.

In order to check whether the reduced two-system density matrix ρ gets entangled at time t because of the time evolution generated by Eq. (3), one can use the *partial transposition* criterion [14,15]: If $\rho(t)$ acted upon with the partial transposition with respect to one of the two subsystems has negative eigenvalues, then it is entangled; in the four-dimensional case we are studying, also the reciprocal is true, namely, if $\rho(t)$ is entangled, then partial transposition makes negative eigenvalues appear.

In physical terms, the bath is not able to create entanglement if and only if the partial transposition preserves the positivity of the state $\rho(t)$ for all times.

Remark 2: Strictly speaking, this criterion allows us to study the possibility of creating entanglement starting from separable initial states. When the initial state is already entangled, the partial transposition criterion cannot settle the question; in such cases, the analysis of the entangling power of the bath can be addressed only through the study of how entanglement measures evolve in time under dissipative reduced dynamics. This problem requires a separate treatment and will not be addressed here.

We therefore take separable states as initial states: as we shall see, this is not really a limitation for the purpose of discussing the possibility of bath-induced entanglement creation. Further, we can restrict our study to pure states; indeed, if the bath cannot create entanglement out of these, it will certainly not entangle their mixtures. In view of this, we will consider initial states of the form

$$\rho(0) = |a_1\rangle\langle a_1| \otimes |b_1\rangle\langle b_1|, \tag{9}$$

where $\{|a_i\rangle\}$, $\{|b_i\rangle\}$, $i = 1, 2$, are orthonormal bases in the two-dimensional Hilbert spaces of the two subsystems. For sake of definiteness, we will operate the partial transposition over the second factor with respect to the basis $\{|b_1\rangle, |b_2\rangle\}$.

One can act with the partial transposition on both sides of Eq. (3) and recast the result as

$$\partial_t \tilde{\rho}(t) = -i[\tilde{H}, \tilde{\rho}(t)] + \tilde{L}[\tilde{\rho}(t)]; \tag{10}$$

here, $\tilde{\rho}(t)$ denotes the partially transposed matrix $\rho(t)$, while $\tilde{H}$ is a new Hamiltonian to which both the unitary and the dissipative term in (3) contribute,

$$\tilde{H} = \sum_{i=1}^{3} H_i^{(1)}(\sigma_i \otimes \mathbf{1}) + \sum_{ij=1}^{3} H_i^{(2)} S_{ij}(\mathbf{1} \otimes \sigma_j) + \sum_{ij=1}^{3} \mathrm{Im}(B \cdot S)_{ij}(\sigma_i \otimes \sigma_j), \tag{11}$$

where S is the diagonal 3×3 matrix given by $S = \mathrm{diag}(-1, 1, -1)$. The additional piece $\tilde{L}[\,\cdot\,]$ is of the form (6), but with a new matrix $\mathcal{D} \to \mathcal{S} \cdot \tilde{\mathcal{D}} \cdot \mathcal{S}$, where

$$\tilde{\mathcal{D}} = \begin{pmatrix} A & \mathrm{Re}(B) + iH^{(12)} \\ \mathrm{Re}(B^T) - iH^{(12)T} & C^T \end{pmatrix}, \tag{12}$$

$$\mathcal{S} = \begin{pmatrix} \mathbf{1}_3 & 0 \\ 0 & S \end{pmatrix}, \tag{13}$$

and the superscript T denotes full transposition, while $H^{(12)}$ is the coefficient matrix in (5).

Remark 3: Although $\tilde{\rho}(t)$ evolves according to a master equation formally of Kossakowski-Lindblad form, the new coefficient matrix $\tilde{\mathcal{D}}$ need not be positive. As a consequence, the time evolution generated by (10) may result to be neither completely positive, nor positive and therefore may not preserve the positivity of the initial state $\tilde{\rho}(0) \equiv \rho(0)$.

Notice that both the Hamiltonian and the dissipative terms of the original master Eq. (3) contribute to the piece $\tilde{L}[\,\cdot\,]$ in (10), the only term in (10) that can produce negative eigenvalues. In particular, this makes more transparent the physical mechanism according to which a direct Hamiltonian coupling $H^{(12)}$ among the two systems can induce entanglement: on $\tilde{\rho}(t)$, $H^{(12)}$ "acts" as a dissipative contribution, which in general does not preserve positivity. The entanglement power of purely Hamiltonian couplings have been extensively studied in the recent literature [1–5]. Instead, in the following we shall concentrate our attention on whether entanglement can be produced by the purely dissipative action of the heat bath; henceforth, we shall ignore the contribution of the matrix $H^{(12)}$ in $\tilde{\mathcal{D}}$. In other words, we shall take into account only baths for which the induced two-system Hamiltonian coupling in (5) is vanishingly small [16].

Remark 4: If $\tilde{D}$ is positive, then the time evolution generated by (10) is completely positive; therefore, $\tilde{\rho}(t)$ is positive at all times and entanglement is not created. Instances of baths for which this happens can easily be provided:

(i) $B = 0$: in such a case, $\tilde{\mathcal{D}}$ is positive since such are A and C^T, due to the positivity of $\mathcal{D}$; this corresponds to a bath that does not dynamically correlate the two subsystems;

(ii) $\mathrm{Re}(B) = 0$: as before, $\tilde{\mathcal{D}}$ is block-diagonal and thus positive;

(iii) $\mathrm{Im}(B) = 0$ and $C^T = C$ or $A^T = A$: in the first case, $\tilde{\mathcal{D}} = \mathcal{D}$, while in the second $\tilde{\mathcal{D}} = \mathcal{D}^T$;

(iv) $A^T = A$ and $C^T = C$: in this case, $\tilde{\mathcal{D}} = (\mathcal{D} + \mathcal{D}^T)/2$.

In the last three cases, despite the fact that the two subsystems are now dynamically correlated by the bath, the effect is not sufficient for entanglement production. Further, notice that entanglement is not created also in baths for which the corresponding coefficient matrix $\mathcal{D}$ can be written as a convex combination of matrices satisfying the previous conditions.

In order to check the presence of negative eigenvalues in $\tilde{\rho}(t)$, instead of examining the full Eq. (10) we find convenient to study the quantity

$$\mathcal{E}(t) = \langle\psi|\tilde{\rho}(t)|\psi\rangle, \tag{14}$$

where ψ is any four-dimensional vector. Assume that an initial separable state $\tilde{\rho}$ has indeed developed a negative eigenvalue at time t, but not before. Then, there exists a vector state $|\psi\rangle$ and a time $t^* < t$ such that $\mathcal{E}(t^*) = 0$, $\mathcal{E}(t) > 0$ for $t < t^*$, and $\mathcal{E}(t) < 0$ for $t > t^*$. The sign of entanglement creation may thus be given by a negative first derivative of $\mathcal{E}(t)$ at $t = t^*$. Moreover, by assumption, the state $\rho(t^*)$ is separable. Without loss of generality, one can set $t^* = 0$ and, as already remarked, restrict the attention to factorized pure initial states.

In other words, the two subsystems, initially prepared in a state $\rho(0) = \tilde{\rho}(0)$ as in (9), will become entangled by the noisy dynamics induced by their independent interaction with the bath if (1) $\mathcal{E}(0) = 0$ and (2) $\partial_t\mathcal{E}(0) < 0$, for a suitable vector $|\psi\rangle$,

$$|\psi\rangle = \sum_{i,j=1}^{2} \psi_{ij}|a_i\rangle \otimes |b_j\rangle. \tag{15}$$

Given (9), condition (1) readily implies $\psi_{11} = 0$.

Remark 5: Note that entanglement creation cannot be detected by looking at the sign of the first derivative of $\mathcal{E}(t)$ unless the test vector $|\psi\rangle$ is entangled itself. Indeed, $\mathcal{E}(t)$ is never negative for a separable $|\psi\rangle$. Thus, both components ψ_{12} and ψ_{21} in (15) have to be different from zero, since otherwise $|\psi\rangle$ becomes separable.

Remark 6: When $\partial_t\mathcal{E}(0) > 0$ for all choices of the initial state $\rho(0)$ and probe vector $|\psi\rangle$, the bath is not able to entangle the two systems, since $\tilde{\rho}$ remains positive. The treatment of the case $\partial_t\mathcal{E}(0) = 0$ requires special care: in order to check entanglement creation, higher order derivatives of $\mathcal{E}$, possibly with a time dependent $|\psi\rangle$, need to be examined.

In order to prove that indeed there are baths for which $\mathcal{E}(0) = 0$ and $\partial_t\mathcal{E}(0)$ is negative, let us first make the choice $|a_1\rangle = |b_1\rangle = |+\rangle$ and $|a_2\rangle = |b_2\rangle = |-\rangle$, where $|\pm\rangle$ are the eigenstates of σ_3; the general case is considered below. For $|\psi\rangle = (|+\rangle \otimes |-\rangle + |-\rangle \otimes |+\rangle)/\sqrt{2}$, one finds

$$\partial_t\mathcal{E}(0) = \mathrm{Tr}[\mathcal{D}\,\mathcal{R}], \tag{16}$$

where $\mathcal{D}$ is as in (7), while

$$\mathcal{R} = \begin{pmatrix} P & Q \\ Q & P \end{pmatrix}, \qquad P = \frac{1}{2}\begin{pmatrix} 1 & i & 0 \\ -i & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{17}$$

and $Q = \mathrm{diag}(-1/2, 1/2, 0)$. Although P is a projector, $(2Q)^2 = \mathrm{diag}(1, 1, 0)$, and as a consequence $\mathcal{R}$ possesses one negative eigenvalue, $(1 - \sqrt{2})/2$, of multiplicity two. Any bath for which the Kossakowski coefficient matrix $\mathcal{D}$ has support only in the negative eigenspace of $\mathcal{R}$ would generate a negative $\partial_t\mathcal{E}(0)$, and therefore entangle the initially separated state $\rho(0) = |+\rangle\langle+| \otimes |+\rangle\langle+|$.

A simple explicit example in which this happens is given by the following two-parameter matrix $\mathcal{D}$, with

$$A = C = \begin{pmatrix} 1 & -ia & 0 \\ ia & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} b & 0 & 0 \\ 0 & -b & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{18}$$

where a and b are real constants [17]. Positivity of $\mathcal{D}$, required by the complete positivity of the subsystem's Markovian dynamics (3), is guaranteed by $a^2 + b^2 \leq 1$. Inside this unit disk, the region for which $\partial_t\mathcal{E}(0)$ in (16) is negative is characterized by the condition $a + b > 1$. Actually, by changing the initial state $\rho(0)$ and the probe vector $|\psi\rangle$, one can show that entanglement is created in all four disk portions outside the embedded square $|a \pm b| \leq 1$. Notice that inside this square $\tilde{\mathcal{D}}$ is positive, so that there the time evolution of the partially transposed density matrix $\tilde{\rho}(t)$ generated by (10) is also completely positive: in this case, entanglement cannot be created for any choice of the initial state $\rho(0)$ and of the vector $|\psi\rangle$.

Now that we have shown that a Markovian dynamics can indeed entangle the two subsystems via a purely noisy mechanism: let us discuss in more detail the condition for entanglement creation. Although in general the basis vectors $|a_i\rangle$, $|b_i\rangle$, introduced in (9), are not eigenstates of σ_3, they can always be unitarily rotated to the basis $|\pm\rangle$:

$$|a_1\rangle = U|+\rangle \quad |a_2\rangle = U|-\rangle, \quad |b_1\rangle = V|+\rangle \quad |b_2\rangle = V|-\rangle. \tag{19}$$

The unitary transformations U and V induce orthogonal transformations $\mathcal{U}$ and $\mathcal{V}$, respectively, on the Pauli matrices:

$$U^\dagger\sigma_i U = \sum_{j=1}^{3} \mathcal{U}_{ij}\sigma_j, \qquad V^\dagger\sigma_i V = \sum_{j=1}^{3} \mathcal{V}_{ij}\sigma_j. \tag{20}$$

With these definitions, for a generic separable initial state (9) and arbitrary vector $|\psi\rangle$ such that $\mathcal{E}(0) = 0$, the condition $\partial_t\mathcal{E}(0) < 0$ for entanglement formation can be expressed as the following expectation value over the product of 6×6 matrices:

$$\vec{w}^{\,\dagger} \cdot [\Psi^\dagger \mathcal{W}^T \tilde{\mathcal{D}} \mathcal{W} \Psi] \cdot \vec{w} < 0, \tag{21}$$

where $\tilde{\mathcal{D}}$ is as in (12) (with $H^{(12)}$ set to zero as explained before), while the remaining matrices are given by

$$\mathcal{W} = \begin{pmatrix} \mathcal{U} & 0 \\ 0 & \mathcal{V} \end{pmatrix}, \qquad \Psi = \begin{pmatrix} \psi_{21}\mathbf{1}_3 & 0 \\ 0 & -\psi_{12}\mathbf{1}_3 \end{pmatrix}, \tag{22}$$

and the components of the 6-vector $\vec{w}$ by the Pauli matrix elements

$$w_i = \langle +|\sigma_i|-\rangle, \qquad w_{i+3} = w_i^*, \qquad i = 1, 2, 3. \tag{23}$$

A more manageable condition for checking entanglement production can be obtained by noticing that (21) is quadratic in the components ψ_{12} and ψ_{21} of $|\psi\rangle$. By suitably rearranging the expression in (21), one can then show that entanglement is generated if the following inequality, independent from the probe vector $|\psi\rangle$, holds:

$$\langle u|A|u\rangle\langle v|C^T|v\rangle < |\langle u|\mathrm{Re}(B)|v\rangle|^2. \tag{24}$$

The 3-vectors $|u\rangle$ and $|v\rangle$ are not completely arbitrary: they contain the information about the starting factorized state (9), and their components can be expressed as

$$u_i = \sum_{j=1}^{3} \mathcal{U}_{ij}\, w_j, \qquad v_i = \sum_{j=1}^{3} \mathcal{V}_{ij}\, w_j^*. \tag{25}$$

Therefore, a given bath will be able to entangle the two subsystems evolving with the Markovian dynamics generated by (3) and characterized by the Kossakowski matrix (7), if there exists an initial state $|a_1\rangle\langle a_1| \otimes |b_1\rangle\langle b_1|$, or equivalently orthogonal transformations $\mathcal{U}$ and $\mathcal{V}$, for which the inequality (24) is satisfied.

The condition (24) can thus be used to check the entangling power of specific Markovian time evolutions. As an example, consider a bath leading to a Kossakowski matrix (7) for which $A = B = C$; this choice corresponds to a special case of collective resonance fluorescence [13,18]. Provided the Hermitian matrix A is not symmetric, one can easily prove that there are initial states of the form (9) with $|a_1\rangle = |b_1\rangle$ that will get entangled by the noisy dynamics. Indeed, in this case condition (24) reduces to

$$|\langle u|\mathrm{Im}(A)|u\rangle|^2 > 0, \tag{26}$$

which is clearly satisfied for any $|u\rangle$ outside the null eigenspace of $\mathrm{Im}(A)$. When A is real, however, (26) is violated and entanglement is not created, since the partial transpose state $\tilde{\rho}(t)$ evolves in time with completely positive dynamics.

The techniques presented here can be applied to other physical settings; a promising one is the Jaynes-Cummings model for two two-level systems [13,19,20], where they can be used to study analytically the possible presence of "collapses" and "revivals" in the entanglement behavior.

Note added.—After completion of the manuscript, our attention was drawn to Refs. [21–24] which have connections with the topics discussed in this Letter.

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