Fermi Gases in Slowly Rotating Traps: Superfluid versus Collisional Hydrodynamics

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The dynamic behavior of a Fermi gas confined in a deformed trap rotating at low angular velocity is investigated in the framework of hydrodynamic theory. The differences exhibited by a normal gas in the collisional regime and a superfluid are discussed. Special emphasis is given to the collective oscillations excited when the deformation of the rotating trap is suddenly removed or when the rotation is suddenly stopped. The presence of vorticity in the normal phase is shown to give rise to precession and beating phenomena which are absent in the superfluid phase.

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Recent experiments on ultracold atomic Fermi gases close to a Feshbach resonance [1–3] have explicitly revealed the emergence of a hydrodynamic regime which shows up in the anisotropic shape of the expanding gas. This behavior is the consequence of the anisotropy of the pressure gradients which are stronger in the tighter directions of the trap. It should be contrasted with the isotropic nature of the expansion of the noninteracting Fermi gas, which instead reflects the initial isotropy of the momentum distribution. In Ref. [4] the anisotropy of the expansion of a Fermi gas has been suggested as a possible signature of superfluidity. The analysis of Ref. [4] concerns, however, dilute gases far from Feshbach resonances where the system, in the normal phase, is collisionless at low temperature due to Pauli blocking [5] and consequently expands differently from a superfluid. If one instead works close to a Feshbach resonance, the huge increase of the collisional rate will favor the achievement of the hydrodynamic regime in the normal phase even at low temperature [1]. At the same time the resonance effect is expected to enhance significantly the value of the critical temperature for superfluidity [6], providing promising perspectives for its experimental realization. Since the anisotropy of the expansion is compatible with either the collisional or superfluid hydrodynamic pictures, its experimental observation cannot be used as a test of superfluidity without further considerations. As a consequence, it is important to identify alternative effects which permit one to distinguish between the two regimes.

The purpose of this Letter is to show that the study of the collective oscillations of a trapped Fermi gas rotating at low angular velocity can provide a useful identification of superfluidity. It is, in fact, well known that a superfluid cannot support vorticity, unless quantized vortices are created. In the following we will consider situations where vortices are absent. This can be ensured by rotating the confining trap at sufficiently low angular velocities. Under these conditions the dynamic behavior of a superfluid is described by the equations of irrotational hydrodynamics:

$$\frac{\partial n}{\partial t} + \boldsymbol{\nabla}\cdot(n\boldsymbol{v}) = 0, \tag{1}$$

$$\frac{\partial \boldsymbol{v}}{\partial t} = -\boldsymbol{\nabla}\left(\frac{v^2}{2}+\frac{V_{\rm ext}}{M}+\frac{\mu}{M}\right) = -\boldsymbol{\nabla}\left(\frac{v^2}{2}+\frac{V_{\rm ext}}{M}\right)-\frac{\boldsymbol{\nabla}P}{Mn}, \tag{2}$$

where $V_{\rm ext}$ is the external potential generated by the confining trap, while P and μ are, respectively, the pressure and the chemical potential of a uniform gas evaluated at the corresponding density. In the last identity we have used the $T = 0$ thermodynamic relationship $\boldsymbol{\nabla}P = n\boldsymbol{\nabla}\mu$. The above equations apply to dynamic phenomena of macroscopic type where the local density approximation to the equation of state is justified. They hold for both Bose and Fermi superfluids at zero temperature. At finite temperature, below T_c, they should be generalized to the equations of two fluid hydrodynamics (see, for example, [7]). Equations (1) and (2) have been systematically used in the last few years to test the effects of superfluidity on the dynamic behavior of Bose-Einstein condensed gases [8].

Differently from a superfluid, a normal gas can support vorticity and in the collisional regime the equation for the velocity field takes the classical Euler form

$$\frac{\partial \boldsymbol{v}}{\partial t} = -\boldsymbol{\nabla}\left(\frac{v^2}{2}+\frac{V_{\rm ext}}{M}\right)-\frac{\boldsymbol{\nabla}P}{Mn}+\boldsymbol{v}\wedge(\boldsymbol{\nabla}\wedge\boldsymbol{v}), \tag{3}$$

where the time dependence of the pressure should be calculated taking into account the conditions of adiabaticity, following from the conservation of entropy. Equation (3) differs from Eq. (2) for the superfluid velocity because of the last term, proportional to the vorticity $\boldsymbol{\nabla}\wedge\boldsymbol{v}$. The hydrodynamic description (3) holds provided the collisional relaxation time τ is much smaller than the inverse of the typical frequencies ω characterizing the dynamic phenomena under investigation, fixed by the oscillator trapping frequencies: $\omega\tau \ll 1$.

Let us first suppose that the initial state of the system does not contain any velocity flow (gas at rest in a static trap). In this case the equations for the expansion as well as for the linearized collective oscillations take the same irrotational form both in the superfluid and

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classical cases. In the following we will assume a trapping potential of harmonic shape: $V_{\rm ext} = M(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2)/2$. If the potential is axially symmetric ($\omega_x = \omega_y = \omega_\perp$), one can classify the solutions for the linearized oscillations in terms of the third component $\hbar m$ of angular momentum. We will consider solutions where the velocity field is linear in the spatial coordinates. The solutions with $m = \pm 2$ and $m = \pm 1$ are surface excitations of the form $\boldsymbol{v} \propto \boldsymbol{\nabla}(x \pm iy)^2$ and $\boldsymbol{v} \propto \boldsymbol{\nabla}[(x \pm iy)z]$ respectively, with frequencies given by

$$\omega(m = \pm 2) = \sqrt{2}\,\omega_\perp, \tag{4}$$

$$\omega(m = \pm 1) = \sqrt{\omega_\perp^2 + \omega_z^2}, \tag{5}$$

independent of the equation of state. Assuming a power law dependence for the isoentropic equation of state ($P \propto n^{\gamma+1}$), one easily finds analytic solutions also for the $m = 0$ modes which are characterized by a velocity field of the form $\boldsymbol{v} \propto \boldsymbol{\nabla}[a(x^2 + y^2) + bz^2]$. The corresponding collective frequencies are given by

$$\omega^2(m = 0) = \tfrac{1}{2}[2(\gamma + 1)\omega_\perp^2 + (\gamma + 2)\omega_z^2 \pm \sqrt{4(\gamma + 1)^2\omega_\perp^4 + (\gamma + 2)^2\omega_z^4 + 4(\gamma^2 - 3\gamma - 2)\omega_\perp^2\omega_z^2}\,]. \tag{6}$$

Equation (6) reduces to the one derived in Ref. [9] in the interacting Bose case ($\gamma = 1$), while in the case of the ideal gas ($\gamma = 2/3$) it reduces to the predictions of Refs. [10–12] and, for spherical trapping, to the ones of Refs. [13]. For elongated traps ($\omega_z \ll \omega_\perp$) one finds $\omega = \sqrt{2(\gamma + 1)}\,\omega_\perp$ and $\omega = \sqrt{(3\gamma + 2)/(\gamma + 1)}\,\omega_z$.

The analysis of the collective frequencies is easily generalized to triaxial anisotropy ($\omega_x \neq \omega_y \neq \omega_z$), where one finds three solutions of the form $\boldsymbol{v} \propto \boldsymbol{\nabla}(xy)$, $\boldsymbol{v} \propto \boldsymbol{\nabla}(xz)$, $\boldsymbol{v} \propto \boldsymbol{\nabla}(yz)$. These are the so-called scissors modes [14] relative to the three pairs of axes, with frequencies $\sqrt{\omega_x^2 + \omega_y^2}$, $\sqrt{\omega_x^2 + \omega_z^2}$, and $\sqrt{\omega_y^2 + \omega_z^2}$, respectively. The other three solutions have the form $\boldsymbol{v} \propto \boldsymbol{\nabla}(ax^2 + by^2 + cz^2)$ and their frequencies obey the equation

$$\omega^6 - (2 + \gamma)(\omega_x^2 + \omega_y^2 + \omega_z^2)\omega^4 + 4(\gamma + 1)(\omega_x^2\omega_y^2 + \omega_x^2\omega_z^2 + \omega_y^2\omega_z^2)\omega^2 - 4(2 + 3\gamma)\omega_x^2\omega_y^2\omega_z^2 = 0. \tag{7}$$

Let us stress again that the above results hold both in the superfluid and normal hydrodynamic phases. In particular, if we excite the scissors mode by suddenly rotating the confining trap starting from the ground state configuration, like in the experiment of Ref. [15], the gas will oscillate with the same frequency in both regimes. The situation would be different in a dilute Fermi gas far from Feshbach resonances where the dynamic response, in the normal phase, is collisionless at low temperature and the behavior of the scissors oscillation exhibits different features with respect to the superfluid case [14,16]. Although in a Fermi gas close to a Feshbach resonance the measurement of the collective frequencies (4)–(7) around the ground state does not provide a direct indication of superfluidity, their experimental determination would be nevertheless very useful, providing an accurate and quantitative proof of the achievement of the hydrodynamic regime. This is best tested looking at the surface excitations, whose frequencies are insensitive to the equation of state.

We are now ready to explore the rotational properties of the system. The most natural procedure is to let a deformed trap rotate at angular velocity Ω in the x-y plane. The angular velocity should be turned on adiabatically in order to ensure the conditions of stationarity. Furthermore, the final angular velocity Ω should be small enough to avoid the formation of quantized vortices in the superfluid phase. This condition is not very restrictive since the angular velocity needed to nucleate vortices by adiabatic increase of the rotation rate is very high [17], as confirmed experimentally in the case of Bose-Einstein condensates [18].

If the system is superfluid, the stationary velocity field has the irrotational form $\boldsymbol{v} = \alpha\boldsymbol{\nabla}(xy)$. The dependence of the coefficient α on the angular velocity Ω has been discussed in Ref. [19] and for small angular velocities reduces to $\alpha = -\epsilon\Omega$, where $\epsilon = (\omega_x^2 - \omega_y^2)/(\omega_x^2 + \omega_y^2)$ is the deformation of the trap in the x-y plane. Actually, in the limit of an axisymmetric trap ($\epsilon = 0$), the velocity field exactly vanishes revealing that the superfluid is not capable to rotate. If, instead, the system is normal, the stationary velocity field takes the rigid form $\boldsymbol{v} = \boldsymbol{\Omega} \wedge \boldsymbol{r}$, corresponding to $\boldsymbol{\nabla} \wedge \boldsymbol{v} = 2\boldsymbol{\Omega}$. Notice that, while the rigid rotation is correctly described by the hydrodynamic Eqs. (1) and (3), the thermalization process, which permits its achievement starting from a gas initially at rest, is not accounted for by Eq. (3), because of the absence of viscosity terms. The time needed to achieve the rigid rotation of the gas (spin-up time) was calculated in Ref. [20], and in the hydrodynamic regime $\omega_\perp\tau \ll 1$ is of the order of $(\epsilon^2\omega_\perp^2\tau)^{-1}$, where τ is the relaxation time fixed by collisions and $\omega_\perp^2 = (\omega_x^2 + \omega_y^2)/2$. This suggests that, in order to reach steady rigid rotation in reasonable times, the system should not be too deeply in the hydrodynamic regime and at the same time the deformation of the rotating trap should not be too small [21].

The fact that the velocity field is so different in the two regimes, gives rise to different predictions for the frequencies of the collective oscillations. For an axisymmetric trap the differences become particularly clear. In fact, while an axisymmetric superfluid cannot rotate and the frequencies of the $m = \pm 2$ quadrupole oscillations are given by Eq. (4), in the collisional hydrodynamic case the degeneracy of these modes is lifted by the presence of the

rigid rotation according to the law [22]:

$$\omega(m = \pm 2) = \sqrt{2\omega_\perp^2 - \Omega^2} \pm \Omega. \quad (8)$$

Equation (8) is consistent with the sum rule approach developed in Ref. [23], yielding the result

$$\Delta\omega = \omega(m = +2) - \omega(m = -2) = \frac{2}{M} \frac{\langle l_z \rangle}{\langle x^2 + y^2 \rangle} \quad (9)$$

for the splitting of the $m = \pm 2$ quadrupole frequencies, where $\langle l_z \rangle$ is the angular momentum per particle. In the case of a superfluid no angular momentum is carried by the system because of the irrotationality constraint, while in the collisional hydrodynamic regime the angular momentum is given by the rigid value $\langle l_z \rangle = M\Omega\langle x^2 + y^2 \rangle$ and hence $\Delta\omega = 2\,\Omega$. Measuring the splitting $\Delta\omega$ then provides an efficient way to detect the vorticity of the gas. The splitting can be measured by suddenly switching off the deformation of the trap. Immediately after, the system will feel the axisymmetric trap $V_{\rm ext} = M[\omega_\perp^2(x^2 + y^2) + \omega_z^2 z^2]/2$, but will no longer be in equilibrium. Actually, in linear approximation the state of the system can be written in the form $|0\rangle + a_+|m = +2\rangle + a_-|m = -2\rangle$, where $|0\rangle$ is the new equilibrium state, while $|m = \pm 2\rangle$ are the $m = \pm 2$ quadrupole states with excitation energies $\hbar\omega_\pm \equiv \omega(m = \pm 2)$. The coefficients $a_\pm$ are fixed by the initial conditions, including the quadrupole deformation and the angular velocity Ω. One finds $a_+ + a_- = \langle x^2 - y^2 \rangle/\sqrt{\sigma}$ and $a_+\omega_+ - a_-\omega_- = 2\,\Omega\langle x^2 - y^2 \rangle/\sqrt{\sigma}$, where σ is the quadrupole strength [23]. The time evolution of the states $|m = \pm 2\rangle$ is fixed by the frequencies $\omega_\pm$ and a simple calculation yields the result

$$\tan(2\theta) = \frac{b_+ \tan(\Delta\omega t/2) + b_- \tan(\omega_Q t)}{b_+ - b_- \tan(\omega_Q t)\tan(\Delta\omega t/2)}, \quad (10)$$

where θ is the angle of the principal axis of the gas in the x-y plane, $\omega_Q = (\omega_+ + \omega_-)/2$, and $b_\pm = a_+ \pm a_-$. In the superfluid case $\Delta\omega = 0$, and one finds the result $\tan(2\theta) = (2\Omega/\omega_Q)\tan(\omega_Q t)$. In the collisional hydrodynamic case, the angle θ exhibits an additional slow precession. This is best seen by taking stroboscopic images at times $t = \pi n/\omega_Q$, with n integer, at which the deformation of the atomic cloud is maximum. For such times $\tan(\omega_Q t) = 0$ and the angle θ, modulus $\pi/2$, follows the precession law $\theta = \Delta\omega t/4$. This precession is caused by the splitting $\Delta\omega = 2\Omega$ and is absent in the superfluid case. The numerical solution of the hydrodynamic equations confirms (see Fig. 1) the accuracy of the prediction (10), based on linear approximation. In the numerical calculation we have chosen the value $\gamma = 2/3$ characterizing the equation of state of an ideal gas, including the most relevant case of a degenerate Fermi gas. The results are not, however, sensitive to the value of γ, consistent with Eq. (10), unless large values of ϵ are considered. The proposed experiment is similar to the one used in [24] to measure the angular momentum of quantized vortices. In that experiment the deformation of the gas was produced by suddenly switching on a laser field in an almost axisymmetric Bose-Einstein condensate. This corresponds to setting $a_+ = a_-$ ($b_- = 0$) and Eq. (10) reduces to $\theta = \Delta\omega t/4$.

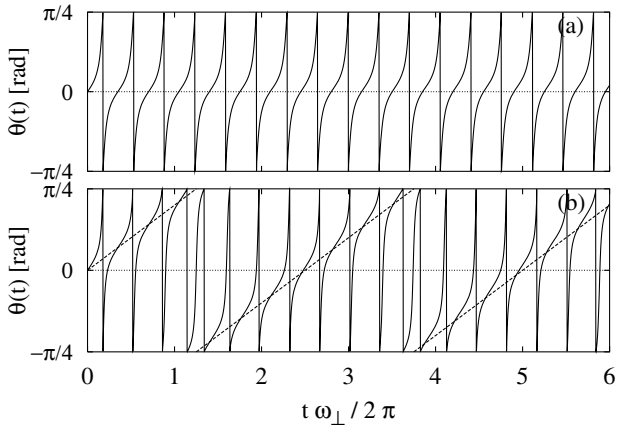

FIG. 1. Time evolution of the angle θ (measured in radians and plotted modulus $\pi/2$) after the sudden switching off of the trap deformation. (a) Superfluid hydrodynamics; (b) collisional hydrodynamics. The trap parameters are $\epsilon = 0.2$, $\Omega = 0.2\omega_\perp$, $\omega_z/\omega_\perp = 0.1$, while $\gamma = 2/3$. The dashed line in (b) is the curve $\theta = \Omega t/2$ (see text).

If instead of switching off the deformation we simply stop the rotation of the deformed trap, another interesting phenomenon takes place that is worth discussing. In fact, the gas, due to its inertia, will first continue rotating, but will soon feel the restoring force produced by the deformed confining potential, generating an oscillation around the new equilibrium configuration. In the superfluid this procedure will excite the usual scissors mode with frequency $\sqrt{2}\,\omega_\perp$. In the case of classical hydrodynamics, the gas will instead oscillate differently. Actually a remarkable property exhibited by Eqs. (1) and (3) of classical hydrodynamics is that they admit stationary solutions with nonvanishing velocity flow also when the trap is at rest in the laboratory frame. These solutions have the form [22] $\boldsymbol{v} = \boldsymbol{\Omega} \wedge \boldsymbol{r} + \alpha\boldsymbol{\nabla}(xy)$ and for small Ω one finds $\alpha = \epsilon\Omega$. Under the condition $\Omega \gg \epsilon^2\omega_\perp$, the resulting oscillation can still be described as a linear combination of the two modes (8) and will consist of the characteristic beating [25]

$$\theta(t) = (\Omega/\omega_Q)\sin(\omega_Q t)\cos(\Delta\omega t/2). \quad (11)$$

Under the same conditions, the intrinsic deformation δ of the cloud also exhibits an oscillatory behavior described by the law $\delta(t) - \delta_0 = -(2\Omega\epsilon/\omega_Q)\sin(\omega_Q t)\sin(\Delta\omega t/2)$, where $\delta_0 = \langle y^2 - x^2 \rangle/\langle x^2 + y^2 \rangle = \epsilon$ is the deformation of the stationary configuration. In Fig. 2 we compare the behavior of the angle θ in the superfluid and normal cases. The numerical results have been obtained by solving the hydrodynamic equations and in the collisional case exhibit the typical beating predicted by Eq. (11).

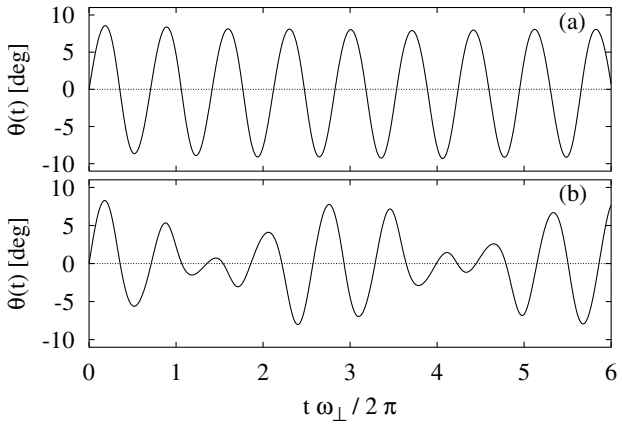

FIG. 2. Time evolution of the angle θ (measured in degrees) after the sudden stop of the rotation of the trap. (a) Superfluid hydrodynamics; (b) collisional hydrodynamics. Parameters as in Fig. 1.

Let us finally recall that the differences between the superfluid and collisional regimes discussed in this Letter concern genuine macroscopic phenomena. The theory of superfluidity predicts also the occurrence of more microscopic quantum phenomena, associated with the quantization of circulation and the appearance of quantized vortices. The description of quantized vortices requires theoretical schemes beyond the hydrodynamic picture and has been the object of recent work in Fermi superfluids in various regimes, including the BCS [26] and the unitarity limit [27]. Their observation, similar to the case of Bose-Einstein condensates [24], could again be revealed by the splitting (9) of the quadrupole frequencies. In a Fermi superfluid, one predicts $\langle l_z \rangle = \hbar/2$ for a single vortex line aligned along the symmetry axis [28]. The realization of quantized vortices, however, requires different procedures with respect to the ones discussed in this Letter. In particular, one should work at higher angular velocity, close to $\omega_\perp/\sqrt{2}$ where the superfluid becomes unstable against the formation of quadrupole deformations [19,29]. Furthermore, one should switch on the rotation of the trap in a nonadiabatic way in order to favor their nucleation. This procedure has already proven to be successful in the experimental realization of quantized vortices in Bose-Einstein condensates [30].

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