Inertial Mass of the Abrikosov Vortex

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We show that a large contribution to the inertial mass of the Abrikosov vortex comes from transversal displacements of the crystal lattice. The corresponding part of the mass per unit length of the vortex line is $M_l = (m_e^2 c^2/64\pi\alpha^2\mu\lambda_L^4)\ln(\lambda_L/\xi)$, where m_e is the bare electron mass, c is the speed of light, $\alpha = e^2/\hbar c \approx 1/137$ is the fine structure constant, μ is the shear modulus of the solid, λ_L is the London penetration length, and ξ is the coherence length. In conventional superconductors, this mass can be comparable to or even greater than the vortex core mass computed by Suhl [Phys. Rev. Lett. **14**, 226 (1965)].

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In this Letter we analyze an old problem of the inertial mass of a moving vortex line in a type-II superconductor. While in most applications of superconductors the dynamics of vortex lines is dissipative, the inertial mass can play an important role at slow motion of vortices. It also enters formulas for quantum tunneling of vortices and formulas for excitations of vortex lines. Besides, it is a fundamental problem of the theory of superconductivity, not without a controversy, repeatedly addressed by a number of researchers over the last 40 years. We show that all previous works on this subject overlooked what appears to be a significant contribution to the vortex mass in conventional type-II superconductors: the torsional deformations of the crystal lattice by the moving vortex. A brief history of the subject is outlined below.

In 1965 Suhl [1] computed two contributions to the vortex mass: the mass due to the kinetic energy of the vortex core, $M_c = (2/\pi^3)mk_F$, and the mass due to the electromagnetic energy of the vortex, $M_{em} = (r_D/\xi)^2 M_c$. Here m is the effective electron mass, k_F is the Fermi wave vector of normal electrons, r_D is the Debye charge-screening length, and ξ is the superconducting coherence length. The vortex core mass arises from the change in the effective mass of electrons by $(\Delta/\epsilon_F)m$ when one moves away from the center of the core, with Δ and ϵ_F being the superconducting gap and the Fermi energy, respectively, (see, e.g., Ref. [2]). As to the electromagnetic mass, it is simply due to the static energy of the magnetic flux carried by the vortex (see, e.g., Ref. [3]). The condition $\xi \gg r_D$ yields $M_c \gg M_{em}$. Suhl's results were later reproduced by other methods (see, e.g., Refs. [2–6]). The main contribution to the vortex mass in neutral superfluids was shown to come from the finite compressibility of the fluid [5,6], which is insignificant in charged superfluids [6]. A separate question, not addressed in this Letter, is the dynamic mass of the vortex in a superclean limit which may be relevant to the low-temperature physics of clean high-temperature superconductors. This mass arises from the quantization of the electron states inside the vortex core, which is important at small ξ. It was shown to exceed the core mass by a large factor $(\epsilon_F/\Delta)^2$ [7–9].

In this Letter we demonstrate that a moving vortex produces torsional shear deformations of the crystal lattice that contribute to the vortex mass. Such deformations are described by the transversal displacement field $\mathbf{u}(\mathbf{r}, t)$ satisfying

$$\nabla \cdot \mathbf{u} = 0. \tag{1}$$

In the long-wave limit they do not affect the density of the ionic lattice. Notice that the contribution to the vortex mass due to longitudinal elastic deformations ($\nabla \times \mathbf{u} = 0$) has been studied before [10–12]. That contribution comes from the tiny difference between the mass density of the normal and superconducting phase and has nothing to do with the effect studied by us. The latter effect arises from the electromagnetic coupling between the superconducting current and the local rotations of the crystal lattice and is a consequence of the momentum conservation [13,14].

We first recall a well-understood problem of a uniform rotation of a superconductor at an angular velocity $\mathbf{\Omega}$ (see, e.g., Refs. [15,16]). In the rotating frame, the electrons feel the effective magnetic field, $\mathbf{H} = (2m_e c/e)\mathbf{\Omega}$. Consequently, due to the Meissner effect, the rotating cylinder develops the magnetic moment $4\pi\mathbf{M} = -(2m_e c/e)\mathbf{\Omega}$, so that the total field in the rotating frame, $\mathbf{H} + 4\pi\mathbf{M}$, is zero. This is a well-known effect predicted by London [17] and first observed by Hildebrandt [18]. Our goal is to study, in the laboratory frame, the effect of local rotations,

$$\mathbf{\Omega}(\mathbf{r}, t) = \tfrac{1}{2}\nabla \times \dot{\mathbf{u}}, \tag{2}$$

caused by the dynamic transverse elastic deformations of the crystal lattice. In the presence of such deformations, the effective vector potential seen by the electrons is

$$\mathbf{A}_{\text{eff}} = \mathbf{A} + \frac{m_e c}{e}\dot{\mathbf{u}}, \tag{3}$$

and the corresponding effective magnetic field is

$$\mathbf{B}_{\text{eff}} \equiv \nabla \times \mathbf{A}_{\text{eff}} = \mathbf{B} + \frac{2m_e c}{e}\mathbf{\Omega}. \tag{4}$$

Notice that $\dot{\mathbf{u}}$ plays a role similar to that of the vector potential $\mathbf{A}$. The second term in Eq. (4) is responsible for the London's magnetic moment of a rotating superconductor discussed above. The essential difference of formulas (3) and (4) from similar formulas for global rotations [15,16] is that they express the local effective vector potential and the local effective magnetic field in terms of the local transversal phonon field. Consequently, the gauge-invariant electric current, expressed in terms of the phase φ of the superfluid wave function, depends on $\mathbf{u}$ through

$$\mathbf{j} = \frac{\hbar e}{2m} n\left(\nabla\varphi - \frac{2e}{\hbar c}\mathbf{A}_{\text{eff}}\right), \tag{5}$$

with m being the effective electron mass and n being the concentration of superconducting electrons. An important observation (needed to explain the value of the London's moment) is that the electrons see the effective vector potential of Eq. (3), with m_e and e being bare electron mass and charge, respectively. The latter is a consequence of the Larmore theorem which provides the relation between $\mathbf{\Omega}$ and $\mathbf{B}$ (or, equivalently, the relation between the mechanical angular momentum and the magnetic moment of electrons) in terms of the bare electron mass and charge regardless of the interactions.

With Eq. (5) in mind, let us study the Lagrangian of the system, $\mathcal{L} = \mathcal{L}_{el} + \mathcal{L}_f$, expressed in terms of coordinates of individual electrons $\mathbf{r}_\alpha$, the phonon displacement field $\mathbf{u}$, and the electromagnetic field $\mathbf{A}$. Here $\mathcal{L}_{el}$ includes all terms that depend explicitly on $\mathbf{r}_\alpha$, while $\mathcal{L}_f$ describes long-wave deformations and electromagnetic fields. Consider now the Raus function [19]

$$R = \sum_a \mathbf{p}_\alpha \dot{\mathbf{r}}_\alpha - \mathcal{L} = \mathcal{H}_{el} - \mathcal{L}_f, \tag{6}$$

which is the Hamiltonian for electrons and (with minus sign) the Lagrangian for $\mathbf{u}$ and $\mathbf{A}$. Here $\mathbf{p}_\alpha = \partial\mathcal{L}/\partial\dot{\mathbf{r}}_\alpha$ are canonical electron momenta. Because electrons adiabatically adjust to the low-frequency long-wavelength displacements of the lattice and the electromagnetic field, one can quantize $\mathcal{H}_{el}$ at any point in a solid as if $\mathbf{A}$ and $\mathbf{u}$ were constant. Since the electrons see the effective vector potential of Eq. (3), the resulting free energy, F, must depend on $\mathbf{u}$ and $\mathbf{A}$ through the combination (3), $F = F[\mathbf{A}_{\text{eff}}]$. The gauge-invariant superconducting current must now be obtained as [20]

$$\mathbf{j} = -c\frac{\delta F[\mathbf{A}_{\text{eff}}]}{\delta\mathbf{A}} = -\frac{e}{m_e}\frac{\delta F[\mathbf{A}_{\text{eff}}]}{\delta\dot{\mathbf{u}}}. \tag{7}$$

Consequently, the effective Lagrangian for $\mathbf{u}$ and $\mathbf{A}$ is

$$\mathcal{L}_{\text{eff}} = -R_{\text{eff}} = \mathcal{L}_f - F[\mathbf{A}_{\text{eff}}] = \mathcal{L}_f + \frac{m_e}{e}\int \mathbf{j}\cdot d\dot{\mathbf{u}}. \tag{8}$$

The part of this Lagrangian that depends on $\mathbf{u}$ determines the effective action for the deformations,

$$S_{\text{eff}}[\mathbf{u}] = \int dt \int d^3r\left[\frac{m_e}{e}\int \mathbf{j}\cdot d\dot{\mathbf{u}} + \frac{1}{2}\rho\dot{\mathbf{u}}^2 - \mu u_{ik}^2\right], \tag{9}$$

where ρ and μ are, respectively, the total mass density and the shear modulus of the solid which, for simplicity, we consider isotropic [21]. The first term in S_{eff} can be traced to the fact that the electric current is determined by the motion of electrons with respect to the ions [14]. The constant μ can be presented as $\mu = \rho c_t^2$, where c_t is the speed of the transversal sound. Notice that the elastic energy of an isotropic solid also contains the term $\frac{1}{2}\lambda u_{kk}^2$ with $\lambda + \frac{2}{3}\mu$ being the compression modulus. In our case this term is zero due to the condition (1). The variation of $S_{\text{eff}}[\mathbf{u}]$ with respect to $\mathbf{u}$ gives the following equation of motion for $\mathbf{u}$:

$$\frac{d^2\mathbf{u}}{dt^2} - c_t^2\nabla^2\mathbf{u} = -\frac{m_e}{e\rho}\frac{d\mathbf{j}}{dt}, \tag{10}$$

which has been derived by Sonin [see Eq. (7) of Ref. [13]], based upon the momentum conservation. This equation is exact as long as the lattice deformations are sufficiently slow to allow for local thermodynamic equilibrium in the electronic subsystem.

According to Eq. (10), a moving vortex produces local shear deformations of the crystal. The way to understand this phenomenon in conventional terms is to notice that the time-dependent magnetic flux of a moving vortex induces the eddy electric field that acts on the ions. We now proceed with the computation of the effective mass of the vortex due to this effect. If the magnetic flux was not quantized, the full treatment of the problem would require a self-consistent solution of the equations of motion for the deformation field $\mathbf{u}$ and the vortex current $\mathbf{j}$. The quantization of the flux, however, makes the vortex a robust source of deformations on the right-hand side of Eq. (10). Let the vortex line be oriented along the Z axis. Then, the current depends on coordinates and time through $\mathbf{j}(\mathbf{r} - \mathbf{V}t)$, where $\mathbf{r}$ is the radius vector perpendicular to Z. For the vortex moving at a speed $V \ll c_t$, the terms in the action and in the equation of motion, that are quadratic on $\dot{\mathbf{u}}$, have a $(V/c_t)^2$ smallness compared to other terms and can be neglected. Integrating the first term in Eq. (9) by parts and omitting small terms, we get

$$S = \int dt \int d^3r\left[\frac{m_e}{e}\mathbf{u}\cdot(\mathbf{V}\cdot\nabla)\mathbf{j} - \mu u_{ik}^2\right]. \tag{11}$$

To compute the vortex mass, we first note that for $\mathbf{V}$ parallel to the X axis

$$\frac{\delta S}{\delta V} = \frac{\delta S}{\delta \mathbf{u}}\frac{\delta \mathbf{u}}{\delta V} + \frac{\partial S}{\partial V} = \frac{\partial S}{\partial V} = \int dt \int d^3 r \frac{m_e}{e} \mathbf{u} \cdot \nabla_x \mathbf{j} \quad (12)$$

for any deformation field $\mathbf{u}(\mathbf{r})$ that satisfies the equation of motion $\delta S/\delta \mathbf{u} = 0$, that is, Eq. (10). In the limit of $V \to 0$, this equation reduces to

$$c_t^2 \nabla^2 \mathbf{u} = -\frac{m_e}{e\rho}(\mathbf{V}\cdot\boldsymbol{\nabla})\mathbf{j}. \quad (13)$$

Its solution must be substituted into Eq. (12), which results in $dS/dV = tZM_l V$, where the vortex mass per unit length, M_l, is expressed in terms of a spatial integral of the second power of the current.

In order to obtain the explicit expression for M_l, we, first, assume that $\mathbf{j}(\mathbf{r})$ is a smooth function of coordinates. Then, the Fourier expansion can be used. Since both $\mathbf{V}$ and $\mathbf{j}$ are perpendicular to Z, it is clear that the spatial Fourier harmonics of $\mathbf{u}$ that couple to $\mathbf{j}$ must have $\mathbf{k}$ and $\mathbf{u}$ also perpendicular to Z. The problem then becomes essentially two dimensional as, of course, is expected from its symmetry. In terms of the Fourier transforms of $\mathbf{j}(\mathbf{r})$ and $\mathbf{u}(\mathbf{r})$ one obtains

$$\frac{dS}{dV} = \frac{im_e}{2e}\int dt \int dz \int \frac{d^2k}{(2\pi)^2} k_x (\mathbf{j_k}\cdot\mathbf{u_k^*} - \mathbf{j_k^*}\cdot\mathbf{u_k}). \quad (14)$$

Substitution of the solution of Eq. (13) into this equation and averaging over the angles yields $S = \int dt \int dz (M_l V^2/2)$, with

$$M_l = \frac{m_e^2}{2e^2\mu}\int d^2 r j^2(r). \quad (15)$$

Note that in the limit of $V \to 0$ no deviation of the current from the solution for a stationary vortex contributes to the mass. In the Landau-Ginzburg theory, the current $j(r)$ of a stationary vortex is given by $j(r) = c\Phi_0/8\pi^2\lambda_L^2 r$ at $\xi < r < \lambda_L$ and by $j(r) \propto \exp(-r/\lambda_L)/\sqrt{r}$ at $r \gg \lambda_L$, where $\Phi_0 = ch/2e$ is the flux quantum, and ξ and λ_L are the coherence length and the London length, respectively [22]. Inside the vortex core ($r < \xi$) the mean-field approximation fails and the current can jump to zero in a nonanalytical manner. In that case special care should be taken to preserve the continuity of the deformation field satisfying Eq. (13). In what follows we obtain the expression for the vortex mass with an account of the possible jump of $\mathbf{j}$ inside the vortex core.

Choosing $\mathbf{j} = j(r)\mathbf{e}_\phi$ and $\mathbf{u} = u_r\mathbf{e}_r + u_\phi\mathbf{e}_\phi$ in the cylindrical reference frame of the vortex, one finds from Eqs. (1) and (13)

$$u_r = G_r(r)\sin\phi, \qquad u_\phi = G_\phi(r)\cos\phi, \quad (16)$$

where $G_r(r)$ and $G_\phi(r)$ satisfy

$$G_\phi = \frac{\partial}{\partial r}(rG_r), \qquad \frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}(rG_r)\right] - G_r = -\frac{Vm_e}{e\mu} r j(r). \quad (17)$$

Substituting $\mathbf{j} = j(r)\mathbf{e}_\phi$ and Eq. (16) into Eq. (12) and performing the angular integration, one obtains

$$\frac{\partial S}{\partial V} = \frac{\pi m_e}{e}\int dt \int dz \int_0^\infty dr\left[\frac{dj}{dr}G_\phi(r) + \frac{j(r)}{r}G_r(r)\right]. \quad (18)$$

Integrating Eq. (18) by parts with the help of Eq. (17), we find

$$M_l = \pi\frac{m_e^2}{e^2\mu}\left\{\int_0^\infty dr r j^2(r) + \xi\Delta\left[j(r)\frac{\partial(rG_r)}{\partial r}\right]\right\}, \quad (19)$$

where we use the notation $\Delta[f(r)] = f(\xi + 0) - f(\xi - 0)$ for the jump of a function $f(r)$ at $r = \xi$.

Equation (19) differs from Eq. (15) by the second term that explicitly depends on the jump in the electric current at the core boundary. The continuity of $\mathbf{u}(\mathbf{r})$ requires the continuity of $G_r(r)$ and $dG_r(r)/dr$. Consequently,

$$\Delta\left[j(r)\frac{d(rG_r)}{dr}\right] = \left[\frac{d(rG_r)}{dr}\right]_{r=\xi}\Delta[j(r)]. \quad (20)$$

For $j(r) = c\Phi_0/8\pi^2\lambda_L^2 r$ at $\xi < r \ll \lambda_L$ and $j = 0$ at $0 < r \le \xi$, the analytical solution of Eq. (17) is $G_r = 0$ at $0 < r \le \xi$ and

$$G_r = \frac{Vm_e}{e\mu}\frac{c\Phi_0}{8\pi^2\lambda_L^2}\left[\frac{1}{4} - \frac{\xi^2}{4r^2} - \frac{1}{2}\ln\left(\frac{r}{\xi}\right)\right] \quad (21)$$

at $\xi \le r \ll \lambda_L$.

According to Eq. (21), the maximal shear deformation of the crystal lattice near the core of a moving vortex is of order $u_{\max} \sim \hbar n V/\mu$, where we have used the relation $\lambda_L^2 = mc^2/4\pi ne^2$ [22]. Substituting here $n \sim 10^{23}\ \text{cm}^{-3}$ and $\mu \sim 10^{11}\ \text{ergs/cm}^3$, one finds that even at $V \sim c_t \sim 10^5$ cm/s $u_{\max}$ is of order 10^{-10} cm; that is, the deformations due to Eq. (13) are always well within the elastic medium approximation. With the help of Eqs. (20) and (21) it is easy to check that the jump term in Eq. (19) is always $\mathcal{O}(1)$ compared to the logarithm, $\ln(\lambda_L/\xi)$, gained by the integration outside the vortex core. We neglect the second term in Eq. (19) when estimating M_l for $\lambda_L \gg \xi$.

We now compute the value of M_l in terms of the measurable parameters of the superconductor. The integral in Eq. (15) can be expressed in terms of the unit-length energy of the vortex line, E_v [22]. In the limit of $\kappa = \lambda_L/\xi \gg 1$,

$$E_v = \frac{2\pi\lambda_L^2}{c^2}\int d^2 r j^2(r) = \left(\frac{\Phi_0}{4\pi\lambda_L}\right)^2 \ln\kappa. \quad (22)$$

This gives for the vortex mass per unit length

$$M_l = \frac{m_e^2c^2}{64\pi\alpha^2\mu\lambda_L^4}\ln\kappa = \frac{\pi}{4}\frac{(\hbar n)^2}{\mu}\left(\frac{m_e}{m}\right)^2\ln\kappa, \quad (23)$$

where we introduced the fine structure constant, $\alpha = e^2/\hbar c \approx 1/137$.

Several observations are in order. First, the nonzero value of M_l is due to the finite rigidity of the crystal with respect to the shear stress. In an absolutely rigid crystal, $\mu = \rho c_t^2 = \infty$ and M_l would be zero. Second, M_l scales as the square of the superfluid density, which is a rather unique feature provided by the specific mass-generating mechanism studied in this Letter. Consequently, M_l should go to zero as $(T_c - T)^2$ when temperature approaches the critical temperature T_c. Third, M_l does not fall into the category of vortex masses [3] that satisfy $M_i \sim E_i/c_i^2$, where E_i is a contribution of some mechanism to the energy of a static vortex and c_i is the velocity of propagation of the distortion in question. This is because no static energy is associated with the mechanism that generates M_l. The novelty of this mechanism is apparent from the explicit dependence of M_l on μ and $\ln\kappa$, which is not present in any previously computed contributions to the vortex mass in charged superfluids.

Equation (23) shows that the vortex mass studied in this Letter is important in metals with a high concentration of superconducting electrons. In copper oxides, because of low n, and also due to strong dynamic effects associated with the quantization of the electron levels in the vortex core [7–9], this mass should not play any significant role. In good metals, however, M_l can be the main contribution to the inertial mass of the vortex line. Taking for estimates $n \sim 10^{23}$ cm^{-3}, $\mu \sim 10^{11}$ ergs/cm^3, $m \sim m_e$, and not very large κ, one gets from the second of Eq. (23), $M_l \sim 10^{-19}$ g/cm. Thus, M_l can easily exceed the inertial mass of the vortex core computed by Suhl, $M_c = (2/\pi^3) m k_F$ [1,2,4], which for the same values of parameters is of the order 10^{-20} g/cm. It is interesting to note that within the model of an isotropic good metal, these two masses, apart from the factor $\ln\kappa > 1$, do not have any parameter smallness with respect to each other. Indeed, taking at $T = 0$, $n = k_F^3/3\pi^2$, $\mu = \rho c_t^2$, $\rho = M_i n/Z$, we get

$$\frac{M_l}{M_c} = \frac{\pi^2}{8}\left(\frac{c_l}{c_t}\right)^2\left(\frac{m_e}{m}\right)^2 \ln\kappa, \tag{24}$$

where we have used the plasma approximation for the speed of the longitudinal sound, $c_l = (Zm/3M_i)^{1/2} v_F > \sqrt{2} c_t$ [21]. However, when using parameters of good metals not very close to T_c, M_l seems always to exceed M_c, as is illustrated by the above numerical example.

Finally, we comment on the contribution of the interaction term in Eq. (9) to the viscosity for the vortex motion. Conservation of linear momentum prohibits the radiation of transversal phonons by a vortex moving at $V < c_t$. At $V > c_t$, however, and especially close to c_t, the viscosity for the vortex motion may, in principle, be dominated by the radiation of the transversal sound rather than by the conventional mechanism due to the finite normal-state resistivity [23]. The full solution of this problem will be presented elsewhere.

In summary, we have computed the contribution to the inertial mass of a moving Abrikosov vortex, which is coming from the torsional deformations of the crystal lattice. A rigorous solution of this problem for an isotropic solid has been obtained. This contribution to the vortex mass can dominate over all other contributions in metals with a high concentration of superconducting electrons.

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[1] H. Suhl, Phys. Rev. Lett. **14**, 226 (1965).
[2] G. Blatter, V. B. Geshkenbein, and V. M. Vinokur, Phys. Rev. Lett. **66**, 3297 (1991).
[3] J.-M. Duan and A. J. Leggett, Phys. Rev. Lett. **68**, 1216 (1992).
[4] M. Yu. Kupriyanov and K. K. Likharev, Zh. Eksp. Teor. Fiz. **68**, 1506 (1975) [Sov. Phys. JETP **41**, 755 (1975)].
[5] J.-M. Duan, Phys. Rev. B **48**, 333 (1993); **49**, 12 381 (1994).
[6] E. B. Sonin, V. B. Geshkenbein, A. van Otterlo, and G. Blatter, Phys. Rev. B **57**, 575 (1998).
[7] N. B. Kopnin, Pis'ma Zh. Eksp. Teor. Fiz. **27**, 417 (1978) [JETP Lett. **27**, 390 (1978)].
[8] A. van Otterlo, M. Feigel'man, V. Geshkenbein, and G. Blatter, Phys. Rev. Lett. **75**, 3736 (1995).
[9] N. B. Kopnin and V. M. Vinokur, Phys. Rev. Lett. **81**, 3952 (1998).
[10] E. Šimánek, Phys. Lett. A **154**, 309 (1991).
[11] M.W. Coffey, Phys. Rev. B **49**, 9774 (1994).
[12] J.-M. Duan and E. Šimánek, Phys. Lett. A **190**, 118 (1994).
[13] E. B. Sonin, Phys. Rev. Lett. **76**, 2794 (1996).
[14] E. M. Chudnovsky and A. B. Kuklov, Phys. Rev. B **67**, 064515 (2003).
[15] B. Cabrera, H. Gutfreund, and W. A. Little, Phys. Rev. B **25**, 6644 (1982).
[16] U. R. Fischer, C. Häussler, J. Oppenländer, and N. Schopohl, Phys. Rev. B **64**, 214509 (2001).
[17] F. London, *Superfluids* (Wiley, New York, 1950), Vol. 1.
[18] A. F. Hildebrandt, Phys. Rev. Lett. **12**, 190 (1964).
[19] L. D. Landau and E. M. Lifshitz, *Mechanics* (Pergamon, London, 1976), Chap. VII, Sec. 41.
[20] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Pergamon, London, 1977), Chap. XV, Sec. 115.
[21] L. D. Landau and E. M. Lifshitz, *Theory of Elasticity* (Pergamon, New York, 1970).
[22] E. M. Lifshitz and L. P. Pitaevskii, *Statistical Mechanics* (Pergamon, London, 1980), Pt. 2, Chap. V.
[23] J. Bardeen and M. J. Stephen, Phys. Rev. **140**, A1197 (1965).