Structure and Kinematics in Dense Free-Surface Granular Flow

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We show that the structure of a dense, free-surface boundary layer granular flow is similar to the structure of a laminar liquid flow: There is a strong component of order (stratification parallel to the mean flow) superposed with a mild component of disorder (self-diffusion perpendicular to the mean flow). We also show that the self-diffusion coefficient scales with the mean velocity and propose a model that relates this scaling to the ordered structure of the flow. Last, we show that the structure of the flow imprints an oscillatory signature (similar to that found in confined granular flow) on the mean velocity profile.

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When a bed of sand is tipped to an angle of about 30 degrees (the angle of repose), the sand in a thin free-surface boundary layer starts to flow down the surface of the bed. This type of granular flow occurs in many industrial processes and natural phenomena [1]. Debris flows, for example, are vast free-surface boundary layer granular flows; in California, they pose a formidable threat to urban areas at the foot of the San Gabriel mountains, whose "loose inimical slopes flout the tolerance of the angle of repose" [2]. Ever since the pioneering work of Bagnold [3], most experimentalists researching granular flow have adopted the Eulerian viewpoint. (In the Eulerian viewpoint, the measurements pertain to fields defined in a fixed control volume [4].) Although the study of Eulerian fields, especially the mean velocity field, has led to many insights into the physics of free-surface granular flow [5] (and also of confined granular flow [6]), much remains to be elucidated. To identify a different path of inquiry into free-surface granular flow, we note that the alternative, Lagrangian viewpoint, is often advantageous, especially when seeking to elucidate the structure of the flow [7]. (In the Lagrangian viewpoint, the measurements pertain to individual flowing particles [4].) Here we start by adopting the Lagrangian viewpoint. In particular, we focus on the particle trajectories and study what they may reveal about the structure of dense, steady, free-surface boundary layer granular flow. Then, we ascertain how the structure of the flow is manifested in aspects of the kinematics other than the particle trajectories. To that end, we study the mean velocity field and the coefficient of self-diffusion perpendicular to the mean flow.

In our experiments we fill a shallow, transparent drum (of diameter 30 cm and depth $2\frac{1}{2}$ bead diameters) halfway with spherical beads (of diameter $d = 2$ or 3 mm). We rotate the drum about its axis with angular velocities between 1 and 5 rpm [Fig. 1(a)]. (For these angular velocities, the boundary layer remains steady and its surface flat.) Then, we focus a digital camera on the center of the boundary layer (where the flow is uniform in the direction of the mean flow) [see Fig. 1(a)] and collect a set of 1024 images at a rate of 500 images per second. Subsequently, we use a computer program [8] to trace the trajectory of each bead throughout the experiment with a resolution of 1/100 of a bead diameter. Figure 1(b) shows part of a typical image from an experiment with 2 mm beads, and the instantaneous velocity vectors of the bead centers (from which we have subtracted the velocity of the drum). The velocity vectors become very small at a depth of seven bead diameters, indicating that the thickness of the boundary layer is about $7d$ in this experiment [9].

To investigate the structure of the flow, we superpose all the trajectories in a single plot. The result (Fig. 2)

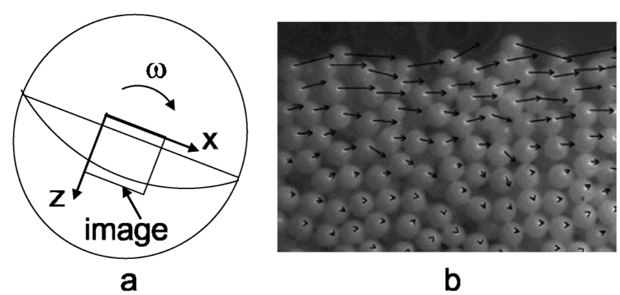

FIG. 1. (a) Schematic of the partially filled rotating drum. The beads flow only within a thin surficial boundary layer (whose thickness is exaggerated in the figure) and primarily parallel to the free surface (from left to right in the x direction). Outside the boundary layer, the beads move in solidlike rotation with the drum. (b) Part of a typical image with computed instantaneous velocity vectors (for an experiment with $d = 2$ mm and $\omega = 1$ rpm). The complete image covers about 3 times as much area as this part. In this and the other images, a bead diameter spans about 40 pixels. We note that the granular flow is dense: Except for the beads located at the free surface, the beads appear always to remain in contact with several neighboring beads.

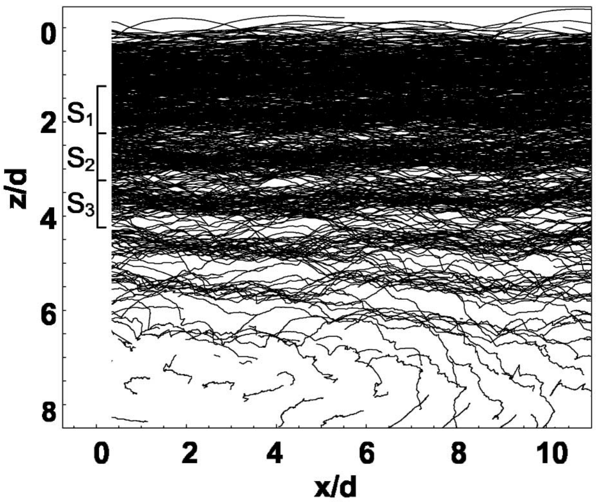

FIG. 2. Bead center trajectories for the same experiment as in Fig. 1(b) above. There is a total of almost 1800 trajectories. The free surface is at the top of the figure ($z \sim 0$) and the beads move from left to right. In the upper portion of the boundary layer ($z < 1.25d$), the beads saltate and the flow is not dense. In most of the boundary layer ($1.25d < z < 7d$), the beads are arranged in strata parallel to the free surface (e.g., S_1, S_2, etc.). Beneath the boundary layer ($z > 7d$) the beads move in circular trajectories, in solidlike rotation with the drum.

shows that the trajectories are grouped in bundles aligned with the direction of the mean flow (the x direction). These bundles define a set of mutually parallel strata in which the probability of finding a bead center is high. (In Fig. 2 we have marked three of these strata with the labels S_1, S_2, and S_3.) The distance between adjacent strata remains close to 1 bead diameter as each stratum slips over the one below it. To verify that the strata are not ephemeral features of the flow, we have performed additional experiments separated by large intervals of time from one another and found the same strata as in Fig. 2.

The stratified structure of the flow revealed by Fig. 2 is similar to the structure of simple laminar liquid flows (e.g., the Poiseuille flow). A flow with the same stratified structure was envisioned by Bagnold [3] in his classic model of the granular flow in a Couette apparatus. In discussing his model, Bagnold noted that "The motions of the grains consist, in addition to a drift in the x-direction, of oscillations in all three directions, involving approaches to, and recessions from, neighbouring grains." This is a fitting description of the motion of the beads *within a single stratum* as they slip over the beads in the stratum below. However, each bead does not persist indefinitely within a single stratum. Instead, Fig. 2 shows that the beads will occasionally jump between adjacent strata. To investigate the excursions of the beads in the direction perpendicular to the strata (the z direction), we perform a statistical analysis of the trajectories. We identify a trajectory by its coordinate pairs (x_i, z_i) measured in the successive images $i = 1, 2$, etc. We consider three sets of trajectories. Each set comprises a number of trajectories $\tau = 1, 2, \ldots, n$ for which the starting point, (x_1^τ, z_1^τ), falls within one of the strata marked S_1, S_2, and S_3 in Fig. 2. For each set of trajectories, we compute the quantities $\Delta x_i^\tau = x_i^\tau - x_1^\tau$, $\Delta z_i^\tau = z_i^\tau - z_1^\tau$, and $(\Delta z_i^\tau)^2 = (z_i^\tau - z_1^\tau)^2$ for all images i and trajectories τ (inset of Fig. 3). Then, we compute the average of these quantities over all the trajectories in the set, i.e., $\langle \Delta x_i \rangle = \sum_{\tau=1}^{n} \Delta x_i^\tau / n$, $\langle \Delta z_i \rangle = \sum_{\tau=1}^{n} \Delta z_i^\tau / n$, and $\langle (\Delta z_i)^2 \rangle = \sum_{\tau=1}^{n} (\Delta z_i^\tau)^2 / n$. Last, we plot $\langle \Delta z_i \rangle / d$ versus $\langle \Delta x_i \rangle / d$ and $\langle (\Delta z_i)^2 \rangle / d^2$ versus $\langle \Delta x_i \rangle / d$ with $i = 1, 2$, etc. (These are parametric plots in the parameter i; we shall omit the subscript i when referring to these plots.) Figure 3 shows these plots for the sets of trajectories S_1, S_2, and S_3.

Suppose that the beads perform random walks in the z direction. Each time a bead moves by one bead diameter in the x direction, there is a probability r that the bead will also move by one bead diameter in the z direction (i.e., that the bead will change stratum). If the bead does move in the z direction, then there is a probability p that the bead will move down, and a probability $1 - p$ that the bead will move up. For a set of many trajectories, these rules lead to the following equations [10]:

$$\langle \Delta z \rangle = r(2p - 1)\langle \Delta x \rangle, \tag{1}$$

and

$$\langle (\Delta z)^2 \rangle = 4rp(1 - p)d\langle \Delta x \rangle. \tag{2}$$

The plots of $\langle \Delta z \rangle / d$ versus $\langle \Delta x \rangle / d$ in Fig. 3 are compatible with (1) if $p = 1/2$. This means that when the beads jump to an adjacent stratum they are just as likely to jump to a deeper stratum as to jump to a less deep stratum.

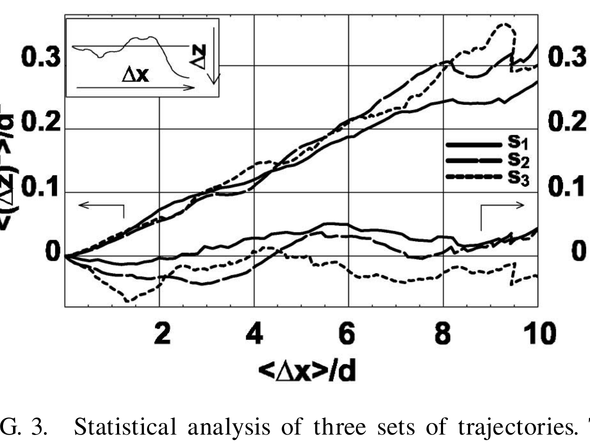

FIG. 3. Statistical analysis of three sets of trajectories. The set S_1, for example, comprises all the trajectories τ for which the starting point, (x_1^τ, z_1^τ), falls within the stratum marked S_1 in Fig. 2; this means that $1.25d < z_1^\tau < 2.25d$, where $2.25 - 1.25 = 1$ is the thickness of the strata in units of the bead diameter. (In the same way, we have $2.25d < z_1^\tau < 3.25d$ for S_2 and $3.25d < z_1^\tau < 4.25d$ for S_3.) The brackets $\langle \cdot \rangle$ denote averaging over all the trajectories in the set. The inset shows a schematic of a single trajectory to illustrate Δx and Δz.

Further, the plots of $\langle(\Delta z)^2\rangle/d^2$ versus $\langle\Delta x\rangle/d$ in Fig. 3 are compatible with (2) if $4rp(1-p) = r = 0.029$. This means that on average the beads jump to an adjacent stratum after having moved a distance of $1/0.029 = 34$ bead diameters in the direction of the mean flow. The visual impression given by Fig. 2 is, therefore, confirmed: The beads move mostly parallel to the free surface and jump between strata only occasionally.

From the previous paragraph, we conclude that the beads perform random walks in the z direction. This is tantamount to concluding that the beads undergo regular self-diffusion perpendicular to the mean flow, just as the atoms (or molecules) do in laminar liquid flows. From the previous paragraph, we can further conclude that the coefficient of self-diffusion in the z direction, D_z, is proportional to the mean velocity in the x direction, $\langle u\rangle$ (where both D_z and $\langle u\rangle$ vary with z). To see this, we note that D_z can be defined by the equation $\langle(\Delta z)^2\rangle = D_z t$ [11]; by comparing this equation with $\langle(\Delta z)^2\rangle = 0.029d\langle\Delta x\rangle = 0.029d\langle u\rangle t$, we obtain $D_z = 0.029d\langle u\rangle$, or $D_z \propto \langle u\rangle$. To explain this result, we propose a "slip-and-shake" model based on the stratified structure of the flow. Let us number the strata starting with 0 for the stratum at the bottom of the boundary layer. (Thus, the mean velocity of stratum 0 is zero, $\langle u_0\rangle = 0$, and the mean velocity of stratum 1 is positive, $\langle u_1\rangle > 0$.) As stratum i slips on the wavy surface of stratum $i-1$, the beads in stratum i (and also the beads in strata $i+1$, $i+2$, etc.) shake in the z direction with a frequency $\Delta\langle u_i\rangle/d$, where $\Delta\langle u_i\rangle$ is the mean velocity of stratum i *relative to stratum* $i-1$, $\Delta\langle u_i\rangle = \langle u_i\rangle - \langle u_{i-1}\rangle$. Because each stratum slips over the one below it, the beads in stratum i shake in the z direction with the concurrent (generally nonharmonic) frequencies $\Delta\langle u_k\rangle/d$ ($k = 1, 2, \ldots, i$); this means that the beads in stratum i jerk in the z direction a number of times per unit time $\sum_{k=1}^{i}\Delta\langle u_k\rangle/d = \langle u_i\rangle/d$. The scaling $D_{zi} \propto \langle u_i\rangle$ follows if we make the sensible assumption that a bead attempts a jump to an adjacent stratum every time it is jerked in the z direction.

It is instructive to compare the results of our statistical analysis of the trajectories with the results of experimental [11] and computational [12] studies of dense granular flow in a Couette apparatus. These studies have shown that in a Couette apparatus the particles undergo regular self-diffusion perpendicular to the mean flow, just as they do in a drum, but with $D_z \propto \langle u\rangle'$ instead of $D_z \propto \langle u\rangle$, where $(\cdot)' = d(\cdot)/dz$. We ascribe this difference in the scaling of D_z to the difference between the boundary conditions of the flow in a drum and the boundary conditions of the flow in a Couette apparatus. In a drum, the surface of the flow is stress free (the flow is driven by gravity) and can accommodate the volumetric fluctuations associated with the slip-and-shake mechanism. In a Couette apparatus, on the other hand, the surface of the flow is *confined* by the walls of concentric cylinders (the flow is driven by a shear stress applied through those walls) and cannot accommodate volumetric fluctuations. We conclude that the scaling $D_z \propto \langle u\rangle$ stems from the two conditions on which we have predicated our slip-and-shake model: the ordered structure of the flow and the free-surface boundary condition.

Having adopted the Lagrangian viewpoint to characterize the structure of the flow, we now turn to the Eulerian viewpoint. We compute two Eulerian fields, the mean volume fraction profile, $\langle f(z)\rangle$, and the mean velocity profile, $\langle u(z)\rangle$, across the boundary layer thickness [13]. Figure 4 shows (a) $\langle f(z)\rangle$ and (b) $\langle u(z)\rangle$ for the same experiment with 2 mm beads of Figs. 1–3. A signature of the stratified structure of the flow is apparent in the mean volume fraction profile, in the form of an oscillation of wavelength equal to the distance between adjacent strata (about $1d$). We discern no such signature in the mean velocity profile; however, a clear signature becomes apparent, again in the form of an oscillation, when we take the derivative of the mean velocity profile

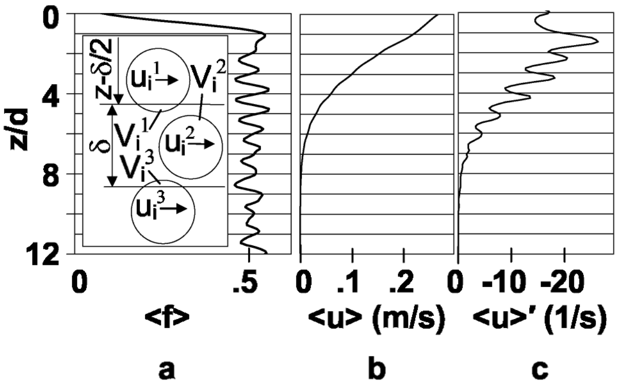

FIG. 4. Eulerian fields for an experiment with $d = 2$ mm and $\omega = 1$ rpm. The inset illustrates the method of computation of the profiles [13]. (a) Mean volume fraction profile [14], (b) mean velocity profile, and (c) derivative of the mean velocity profile with respect to z.

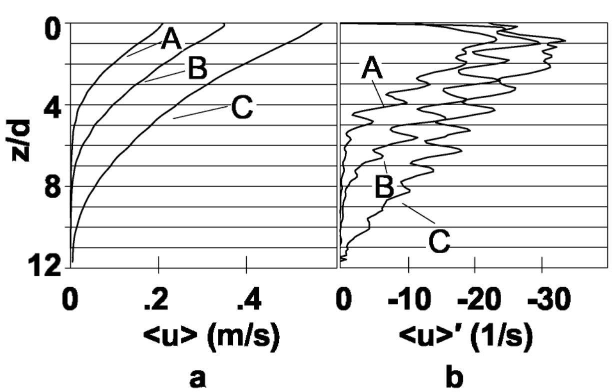

FIG. 5. Eulerian fields for experiments with $d = 3$ mm and $\omega = 1$ rpm (curves marked A), 3 rpm (curves marked B), and 5 rpm (curves marked C). (a) Mean velocity profiles and (b) derivatives of the mean velocity profiles with respect to z.

with respect to z [Fig. 4(c)]. We conclude that the stratified structure of the flow imprints a distinct oscillatory signature on the Eulerian fields [15]. We confirm this conclusion in experiments with 3 mm beads and three different angular velocities of the drum (Fig. 5; note that in all cases the oscillations are of wavelength $1\,d$).

Mueth *et al.* [6] documented oscillations similar to those of Figs. 4 and 5 in experiments with dense granular flow in a Couette apparatus. (In addition, they showed that the oscillations are less pronounced when the particles are less perfectly spherical or less smooth.) In earlier work, Savage and Dai [12] documented an oscillation similar to that of Fig. 4(a) in a computational simulation of dense granular flow in a Couette apparatus. (In addition, they showed that the oscillation occurs only when the flow is dense.) Based on these results, Mueth *et al.* and Savage and Dai surmised that the dense granular flow in a Couette apparatus must be stratified in the way envisioned by Bagnold [3]. Thus, in spite of the difference in boundary conditions (and of the attendant difference in the self-diffusion coefficient), the confined granular flow in a Couette apparatus and the free-surface granular flow in a partially filled drum appear to be similarly structured.

We have characterized the simplest possible laminar-like structure in a dense, free-surface boundary layer granular flow. Granular flows with more complex laminarlike structures appear to be possible at higher velocities; for example, Forterre and Pouliquen have reported a dense, free-surface boundary layer granular flow with vortices parallel to the direction of the flow [16]. It remains to be ascertained whether a turbulentlike structure would occur at still higher velocities, as a recent computational simulation suggests [17].

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[9] Although we ascribe a "finite" thickness to the boundary layer, the velocity vectors may not vanish at a finite depth; instead, the recent experimental results of Komatsu *et al.* indicate that the velocity vectors vanish asymptotically with depth, just as they do in the viscous (Prandtl-Blasius) boundary layers of liquid flows. See T. S. Komatsu, S. Inagaki, N. Nakagawa, and S. Nasuno, Phys. Rev. Lett. **86**, 1757 (2001).

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[13] To compute the profiles, we partition each of the 1024 images from one experiment into equally sized bins parallel to the x direction and of thickness δ in the z direction. Consider the bin centered at a depth z , B_z [inset of Fig. 4(a)]. Associated with B_z, we compute one point of the mean velocity profile, $\langle u(z)\rangle = \langle u\rangle_{B_z} = (\sum_i \sum_b u_i^b V_i^b)/\sum_i \sum_b V_i^b$ and one point of the mean volume fraction profile, $\langle f(z)\rangle = \langle f\rangle_{B_z} = \sum_i \sum_b V_i^b/1024V$. In these expressions, the sum in i extends over all the 1024 images, the sum in b extends over all the beads, u_i^b is the instantaneous velocity of the bead b in the image i, V_i^b is the portion of the volume of the bead b which falls within B_z in the image i, and V is the volume of B_z, $V = w\delta d$, where w is the width of the bin in the x direction. The profiles obtained in this way become invariant to changes in δ when $\delta < d/10$. We use $\delta = d/20$.

[14] For comparison, the volume fraction of a square packing is 0.524, and that of a hexagonal packing is 0.605.

[15] The detection of analogous oscillatory signatures in the volume fraction and velocity profiles of laminar liquid flows would require tracking the individual atoms (or molecules) of the liquid within a length scale comparable to the size of the atoms. This would appear to be experimentally unfeasible.

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