Ordering Chaos by Random Shortcuts

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In this Letter, the effects of random shortcuts in an array of coupled nonlinear chaotic pendulums and their ability to control the dynamical behavior of the system are investigated. We show that random shortcuts can induce periodic synchronized spatiotemporal motions, even though all oscillators are chaotic when uncoupled. This process exhibits a nonmonotonic dependence on the density of shortcuts. Specifically, there is an optimal amount of random shortcuts, which can induce the most ordered motion characterized by the largest order parameter that is introduced to measure the spatiotemporal order. Our results imply that topological randomness can tame spatiotemporal chaos.

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Over the last two decades, counterintuitive phenomena induced by noise and disorder have attracted increasing attention. It is now well known that intrinsically noisy and disordered processes, such as thermal fluctuations or mechanically randomized scattering, can generate surprisingly ordered patterns in nonlinear systems [1]. Intriguing examples include stochastic resonance (SR) [2], noise-induced transition [3], noise-sustained waves [4], spatiotemporal chaos tamed by disorder [5], disorder-enhanced synchronization [6], etc. But understanding the emergence of order induced by different types of randomness in these complex systems is a formidable challenge to statistical mechanics. Presently, complex networks have attracted considerable interests, the main reason being that they seem to be exceedingly simple model systems of complex behavior in real world systems [7]. In fact, any complex system in nature can be modeled as a network, where vertices are the dynamic elements of the system and the edges represent the interactions between them. The dominant study on complex networks so far is to investigate the topological properties of the networks and various mechanisms that determine the topology. Many models were presented, including small world networks and scale-free networks [8]. It has been shown that, in many real-life cases, connections among network elements are neither completely random nor completely local (regular), but somewhere in between. In other words, real networks have some degree of *topological randomness*. It is well accepted that the topology of a network often plays crucial roles in determining the dynamic features of the system. Therefore, it is natural to ask if *this new type of randomness might play some constructive roles for the system's dynamic features, like noise and disorder.* To our knowledge, few investigations focus on this topic so far [9]. This Letter provides one intriguing example and gives a positive answer to this question.

In this Letter, we have studied the collective dynamic behavior of an array of coupled chaotic oscillators with direct shortcuts between randomly chosen oscillators. Coupled oscillators provide a simple but powerful mathematical model for simulating the collective behavior of a variety of systems that are of interest in physics, chemical, and biological sciences [10]. In general, there could be a wide variety of collective behaviors, such as phase synchronization, phase trapping, spatiotemporal chaos, and so on. In this study, nontrivial effects of random shortcuts have been found. On one hand, we find that there exists an optimal fraction of shortcuts that can tame the spatiotemporal chaos observed in the regular array to periodic synchronized motion. On the other hand, addition of a little randomness to a regular network can lead to synchronization more effectively than regular networks or completely random networks.

We consider an array of forced, damped, pendulums governed by the following equation:

$$ml_n^2\ddot{\theta}_n + \gamma\dot{\theta}_n = -mgl_n\sin\theta_n + \tau' + \tau\sin\omega t + \sum_m \kappa_{nm}(\theta_m - \theta_n), \quad (1)$$

where $n = 0, 1, 2, \ldots, N-1$, $N = 128$, and the boundary condition is free ($\kappa_{nm} = 0$ if $m < 0$ or $m > N-1$). The parameters used are the gravitational acceleration $g = 1.0$, mass of the pendulum $m = 1.0$, length $l_n = 1.0$, dc torque $\tau' = 0.7155$, ac torque $\tau = 0.4$, the angular frequency $\omega = 0.25$, and the damping $\gamma = 0.75$. κ_{nm} is the coupling strength between the two oscillators n and m, which is determined by the coupling pattern of the system. If these two oscillators are coupled to each other, we have $\kappa_{nm} = \kappa = 0.5$, and otherwise $\kappa_{nm} = 0$. We numerically integrate Eq. (1) using a fourth order Runge-Kutta technique with a time step $dt = 0.001$. For an isolated pendulum, the dynamic behavior is chaotic for the default length $l = 1.0$, which is characterized by a positive Lyapunov exponent. For $l > 1.0$, the pendulum executes a libration in which it oscillates about its equilibrium position without overturning; i.e., the angle θ never exceeds 2π. If $l < 1.0$, the pendulum executes a whirling where the combined torques rotate the pendulum over the top and the angle θ past 2π.

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To add randomness to the topological structure of the network, we start from a regular array where each site is connected to its two nearest neighbors. Then we randomly add links between non-nearest sites. The number of random shortcuts is denoted by M, and the fraction of random shortcuts, which is the ratio of random shortcuts to all possible number of edges among the oscillators, is given by $q = 2M/(N-1)(N-2)$.

Figure 1 describes the spatiotemporal evolution of an array of 128 pendulums. Time passes from the bottom to top. The colors code the angular velocities of each pendulum: black denotes negative velocities and white denotes positive ones. The narrow strips of black and white represent sudden motion of the oscillators. In the absence of random shortcuts ($q = 0$), spatiotemporal chaos is observed as shown in Fig. 1(a). However, when a certain number of random shortcuts are present, we find that the system shows a very regular spatiotemporal pattern, which is synchronized in space and periodic in time. Such an example is depicted in Fig. 1(b) for $q = 0.01$ and $M \sim 80$, where the synchronized pattern repeats every four forcing periods. If the number of random shortcuts is further increased, the motion of the array is still synchronized in space, but now chaotic in time. This observation demonstrates the phenomenon of "ordering chaos by random shortcuts" and the existence of an optimal level of topological randomness such that the spatiotemporal evolution of the system is the most ordered.

To further characterize this behavior quantitatively, we introduce a quantity to measure the regularity of the spatiotemporal pattern. It is based on the normalized autocorrelation function $c_i(\tau_d)$, defined as $c_i(\tau_d) = \langle \tilde{\dot{\theta}}_i(t)\tilde{\dot{\theta}}_i(t+\tau_d)\rangle / \langle \tilde{\dot{\theta}}_i^2\rangle$, where $\dot{\theta}_i(t)$ is the angular velocity of the ith pendulum at time t, τ_d is the time delay, $\tilde{\dot{\theta}}_i(t) = \dot{\theta}_i(t) - \langle\dot{\theta}_i\rangle$, and the averaging is taken over the time. A characteristic correlation time for the ith pendulum is then evaluated as $\tau_{i,c} = \frac{1}{T}\int_T c_i^2(t)dt$, following Pikovsky *et al.* [11]. In the present case of limited and discrete sampling with N_0 data points for each oscillator, the characteristic correlation time is given by $\tau_{i,c} = \frac{1}{N_0\Delta t}\sum_{k=1}^{N} c_i^2(\tau_k)\Delta t$, where $\tau_k = k\Delta t$ with Δt being the sampling time, and $N_0\Delta t$ being the length of the time series.

Then the "order parameter" for given q is defined as $\tau(q) = [\langle\tau_{i,c}\rangle]$, where angular brackets denote the averaging over all the pendulums and square brackets the averaging over 100 different network realizations with the same q. The more ordered a pendulum oscillation is, the longer is its characteristic correlation time and hence its contribution to the order parameter. Therefore, this quantity can be readily used to measure the degree of spatiotemporal order in the present system [12].

The dependence of this quantity on the fraction of random shortcuts q is presented in Fig. 2. It has a clear maximum around $q \approx 0.01$, where all pendulums run periodically with the same phase. This gives an evident example that the system dynamics show somewhat "resonant" behavior with an optimal level of topological randomness, similar to the effects of noise and disorder in nonlinear systems such as SR. One should note that this resonant behavior is nontrivial. To establish the importance of random connections, we have also generated results for regular array with nonrandom long-range connections and for completely random arrays. First,

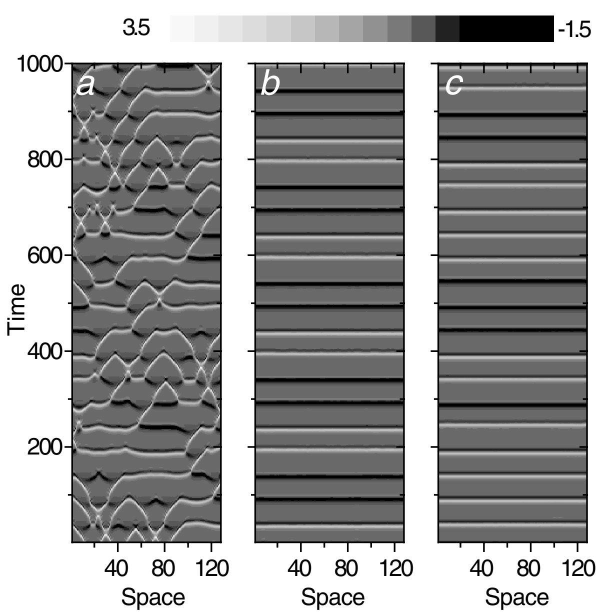

FIG. 1. Spatiotemporal evolution of a chain of 128 coupled pendulums with random shortcuts. Time increases from bottom to top. From the left to right, q is 0.0, 0.01, and 0.02.

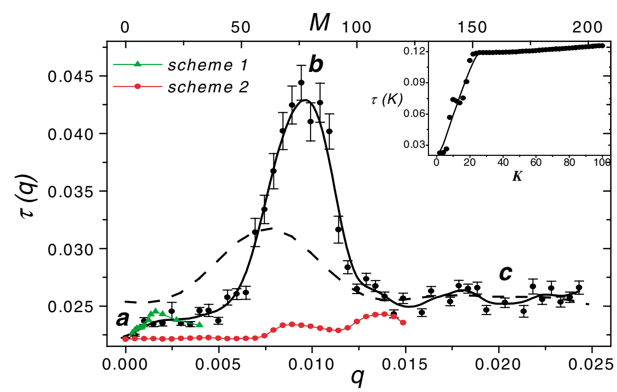

FIG. 2 (color online). Dependence of $\tau(q)$ on q (or M) in the regular array with random shortcuts (circle with error bar), the completely random network (dashed line), and two schemes of adding regular long-range connections on regular array. The curve with symbol ▲ and the curve with symbol ● correspond to scheme 1 and scheme 2, respectively. The inset shows the dependence of τ on K (the number of connections per site) in a regular array (see text). Spatiotemporal evolution pattern for regimes (a), (b), (c) are shown in Fig. 1.

for a regular array with each site connecting to its K neighbors (i.e., site i is connected to site $i - K/2, \ldots, i - 1, i + 1, \ldots, i + K/2$), the order parameter τ defined above increases sharply at first and then nearly saturates, as shown in the inset of Fig. 2. However, one notes that, for $K = 4$, there are already 126 additional edges added to the original regular array with $N - 1 = 127$ edges, and the corresponding order parameter τ is only 0.021, which is much smaller than the value of $\tau = 0.045$ for $q \approx 0.01$, where the amount of shortcuts are only about 80. Second, we consider two schemes to create regular long-range connections only for a subset of the cells. In scheme 1, we add connections between site i and $i + k$ if $(i \bmod 2k) = 0$. For $k = 4$, this would create links between elements 0 and 4, 8 and 12, etc. In scheme 2, connections are added between the elements i and $N - M + i$, $i = 1, 2, \ldots, M$. Notice that for both schemes the maximum number of nonrandom shortcuts M is limited by the system size N. The dependence of the order parameter on the number of nonrandom shortcuts for these two schemes is also shown in Fig. 2. It can be seen that no nontrivial behavior appears for both schemes. From this point of view, the addition of random shortcuts to a regular array is more effective to achieve a periodic spatiotemporal state than regularly adding long-range connections. Finally, we have also studied the case of a completely random array, where $M + N - 1$ edges are randomly distributed to the N elements. In this situation, we find that τ also has a maximum with the increment of M (see also Fig. 2), but the peak value is much smaller. Therefore, the sharp peak in the $\tau - q$ curve is the combined effect of regular network and random shortcuts. Neither regular networks only nor completely random networks only can induce such a nontrivial phenomenon.

From the simulation results above, one may understand the mechanism qualitatively, though we have not performed an analytical explanation. It is reasonable that enough long-range connections can synchronize the pendulum array. On one hand, just a few such shortcuts can create local structures, with effective average pendulum lengths significantly different from unity, that entrain the entire array in periodic motion, either rotation/whirling or libration. Therefore, the spatiotemporal chaos is tamed and a synchronized periodic motion is observed. On the other hand, if adding more long-range shortcuts, local structures will be smoothed such that the whole system behaves like a single pendulum and a synchronized chaotic motion results. Since a regular array is necessary for local structure and long-range shortcuts are necessary for synchronization, one can conclude that the nonmonotonic behavior is a combined effect of both regular array and random long-range shortcuts. The local maximum in the order parameter for the random array might be explained as follows: since we have had a lot of realizations when generating a random array, there is some chance that some realizations of the random array can be viewed as "regular array" plus "random shortcuts." Therefore, a local maximum also exists in the order parameter, but the value is much smaller. Alternatively, we may understand this phenomenon from another point of view. Generally, a spatially extended system would occupy a large high-dimensional parameter space. Every neighborhood in the parameter space will be associated with lots of dynamic attractors. It is possible that adding random shortcuts will shift the system's dynamics from the spatiotemporal chaotic attractor to a synchronized periodic attractor nearby [5].

To summarize, we have studied the collective dynamical behavior of an array of coupled pendulums with a small fraction of random long-range connections. We show that the spatiotemporal chaos observed in a regular way can be tamed into synchronized periodic motion. In addition, there is an optimal amount of random shortcuts, which can induce the most periodically synchronized spatiotemporal motions. It is strongly against the intuition since it is generally accepted that random connections are not favorable to the formation of any regular spatiotemporal patterns. It also gives a novel example describing how order can emerge from systems with topological randomness. The collective behaviors arise not only on the element's intrinsic dynamical processes, but also on their coupling patterns between each other. Since control and synchronization of chaotic dynamics have been established as a central topic in nonlinear science [13–15], we are sure that the proposed approach here should have potential applications in such systems as Josephson array or semiconductor laser arrays, where any type of regular behavior is preferred to chaos. The effect of shortcuts may be utilized as a new efficient strategy for the control of other discrete and continuous dynamical systems, such as coupled map lattices, turbulence, spatiotemporal chaos, etc.

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