Theory of an Entanglement Laser

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We consider the creation of polarization entangled light from parametric down-conversion driven by an intense pulsed pump field inside a cavity. The multiphoton states produced are close approximations to singlet states of two very large spins. A criterion is derived to quantify the entanglement of such states. We study the dynamics of the system in the presence of losses and other imperfections, concluding that the creation of strongly entangled states with photon numbers up to a million seems achievable.

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Entanglement of light has first been demonstrated at the few-photon level, both for polarization entanglement in the context of Bell inequality experiments [1] and for continuous-variable EPR-type entanglement [2]. It is a challenging goal to extend these results towards the domain of macroscopic light. Continuous-variable entanglement for intense fields has recently been demonstrated in Refs. [3], and polarization entanglement of macroscopic beams in Ref. [4]. In this latter experiment the quantum fluctuations around two macroscopic polarized beams are entangled.

Here we consider polarization entangled states of a different kind. We aim to create entangled pairs of light pulses such that the polarization of each pulse is completely undetermined, but the polarizations of the two pulses are always anticorrelated. Such a state is the polarization equivalent of an approximate singlet state of two very large spins. It is thus a dramatic manifestation of multiphoton entanglement.

We propose a scheme that is based on the nonlinear optical effect of parametric down-conversion driven by a strong pump pulse, where the interaction length is increased by cavities for both the pump and the down-converted light. Starting from a spontaneous process, the proposed setup builds up entangled states which have very large photon populations per mode, corresponding to strong stimulated emission, and thus deserves the name of an "entanglement laser." The basic principle of stimulated entanglement creation was experimentally demonstrated in the few-photon regime in Ref. [5].

Multiphoton entanglement of the kind under consideration has been theoretically studied in the context of Bell's inequalities [6]. The violation of the inequalities for large photon numbers studied in that work is very sensitive to photon loss. This leads to the question whether the entanglement itself is also very fragile. To analyze whether multiphoton polarization entanglement can be generated in the presence of losses and other imperfections, we derive a simple inseparability criterion that is formulated in terms of the total spin $\mathbf{J}$ and the total photon number N: if $\langle \mathbf{J}^2 \rangle / \langle N \rangle$ is smaller than 1/2, then the state is entangled. Using this criterion we show that the entanglement is quite robust, and that strongly entangled states of very high photon numbers can be generated under realistic conditions.

Let us now study our system in more detail. The source of entangled light [7] is described by a Hamiltonian

$$H = i\kappa(a_h^\dagger b_v^\dagger - a_v^\dagger b_h^\dagger) + \text{H.c.}, \tag{1}$$

where a and b refer to the two conjugate directions along which the photon pairs are emitted, as shown in Fig. 1, h

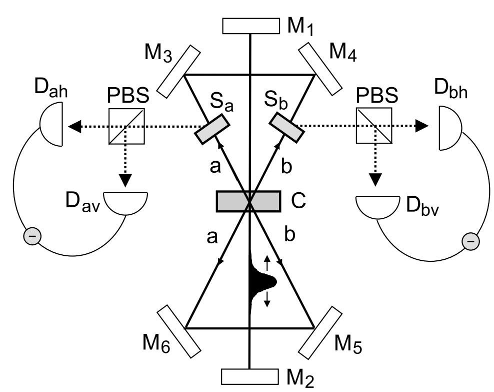

FIG. 1. Proposed setup for an "entanglement laser." An intense pump pulse propagates back and forth between the mirrors M_1 and M_2. Whenever it traverses the nonlinear crystal C it creates polarization entangled photon pairs into the modes a and b, which are counterpropagating pulses inside the cavity formed by the mirrors M_3 to M_6. The interferometrically stable cavities are adjusted such that the three pulses (pump, a and b) always overlap in the crystal. The number of photons in a and b increases exponentially with the number of round-trips. They can be switched out of the cavity by electro-optic switches S_a and S_b. The polarization of each pulse is then analyzed with the help of polarizing beam splitters (PBS) followed by photodiodes that give a signal proportional to the number of photons. Taking the difference between the photon numbers for the two polarizations behind each PBS corresponds to a spin measurement. The axis of spin analysis is changed by appropriate wave plates in front of the PBS.

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and v denote horizontal and vertical polarization, and κ is a coupling constant whose magnitude depends on the nonlinear coefficient of the crystal and on the intensity of the pump pulse. The Hamiltonian describes two phase-coherent twin beam sources, corresponding to the pairs of modes a_h, b_v and a_v, b_h. In the absence of losses, it produces a state of the form

$$|\psi\rangle = e^{-iHt}|0\rangle = \frac{1}{\cosh^2\tau}\sum_{n=0}^{\infty}\sqrt{n+1}\tanh^n\tau|\psi_-^n\rangle, \quad (2)$$

where $\tau = \kappa t$ is the effective interaction time and

$$|\psi_-^n\rangle = \frac{1}{\sqrt{n+1}}\frac{1}{n!}(a_h^\dagger b_v^\dagger - a_v^\dagger b_h^\dagger)^n|0\rangle = \frac{1}{\sqrt{n+1}}\sum_{m=0}^{n}(-1)^m|n-m\rangle_{a_h}|m\rangle_{a_v}|m\rangle_{b_h}|n-m\rangle_{b_v}. \quad (3)$$

All terms in the expansion in Eq. (3) have the same magnitude, such that the observed polarization (the difference in the number of horizontal and vertical photons) will fluctuate strongly. However, there is a perfect anticorrelation between the a and b pulses. The state $|\psi\rangle$ looks the same if the axis of polarization analysis is rotated by the same amount for the a and b modes. It is the polarization equivalent of a spin singlet state [8], where the spin components correspond to the Stokes parameters of polarization, $J_z^A = \frac{1}{2}(a_h^\dagger a_h - a_v^\dagger a_v)$, $J_x^A = \frac{1}{2}(a_+^\dagger a_+ - a_-^\dagger a_-)$, and $J_y^A = \frac{1}{2}(a_l^\dagger a_l - a_r^\dagger a_r)$. The spin components can thus be expressed as differences in photon numbers, where $a_{+,-} = (1/\sqrt{2})(a_h \pm a_v)$ correspond to linearly polarized light at $\pm 45°$, and $a_{l,r} = (1/\sqrt{2})(a_h \pm ia_v)$ to left- and right-handed circularly polarized light. The label A refers to the a modes; cf. Fig. 1. Analogous relations express $\mathbf{J}^B$ in terms of the b modes. The total spin satisfies $(\mathbf{J}^A)^2 = (J_x^A)^2 + (J_y^A)^2 + (J_z^A)^2 = \frac{N_A}{2}(\frac{N_A}{2}+1)$. Number states of the modes a_h and a_v are eigenstates of J_z^A and of $(\mathbf{J}^A)^2$. The state $|n-k\rangle_{a_h}|k\rangle_{a_v}$ has total spin $j = n/2$ and J_z^A eigenvalue $m = (n-2k)/2$.

The states $|\psi_-^n\rangle$ of Eq. (3) are singlet states of the total angular momentum operator $\mathbf{J} = \mathbf{J}^A + \mathbf{J}^B$ for fixed $j_A = j_B = n/2$. As a consequence, $\langle\psi|\mathbf{J}^2|\psi\rangle = 0$ also for the state $|\psi\rangle$ of Eq. (2). Losses and imperfections lead to nonzero values for the total angular momentum, corresponding to nonperfect correlations between the Stokes parameters in the a and b pulses. Since the ideal state of Eq. (2) is highly entangled, one expects that states in its vicinity are still entangled. We now present a convenient criterion for entanglement: for *separable* states

$$\frac{\langle\mathbf{J}^2\rangle}{\langle N\rangle} \geq \frac{1}{2}, \quad (4)$$

where $\mathbf{J} = \mathbf{J}^A + \mathbf{J}^B$ and $N = N_A + N_B$. To prove this, consider $\langle\mathbf{J}^2\rangle$ for a separable state $\rho = \sum_i p_i\rho_i^A \otimes \sigma_i^B$. One has

$$\langle\mathbf{J}^2\rangle = \langle(\mathbf{J}^A)^2\rangle + \langle(\mathbf{J}^B)^2\rangle + 2\langle\mathbf{J}^A\cdot\mathbf{J}^B\rangle = \sum_i p_i\langle(\mathbf{J}^A)^2\rangle_i + \sum_i p_i\langle(\mathbf{J}^B)^2\rangle_i + 2\sum_i p_i\langle\mathbf{J}^A\rangle_i\langle\mathbf{J}^B\rangle_i \geq \sum_i p_i[\langle(\mathbf{J}^A)^2\rangle_i + \langle(\mathbf{J}^B)^2\rangle_i - 2|\langle\mathbf{J}^A\rangle_i||\langle\mathbf{J}^B\rangle_i|] \geq \sum_i p_i[\langle(\mathbf{J}^A)^2\rangle_i + \langle(\mathbf{J}^B)^2\rangle_i - 2\alpha_i\beta_i], \quad (5)$$

where $\langle\mathbf{J}^A\rangle_i = \mathrm{Tr}\rho_i^A\mathbf{J}^A$, $\langle\mathbf{J}^B\rangle_i = \mathrm{Tr}\sigma_i^B\mathbf{J}^B$, etc. Furthermore, $\alpha_i = \sqrt{\langle(\mathbf{J}^A)^2\rangle_i + \frac{1}{4}} - \frac{1}{2}$, $\beta_i = \sqrt{\langle(\mathbf{J}^B)^2\rangle_i + \frac{1}{4}} - \frac{1}{2}$, and we have used the fact [9] that $|\langle\mathbf{J}\rangle| \leq \sqrt{\langle\mathbf{J}^2\rangle + \frac{1}{4}} - \frac{1}{2}$. The last line of Eq. (5) can be rewritten as

$$\sum_i p_i[\alpha_i^2 + \alpha_i + \beta_i^2 + \beta_i - 2\alpha_i\beta_i] = \sum_i p_i[(\alpha_i - \beta_i)^2 + \alpha_i + \beta_i] \geq \sum_i p_i[\alpha_i + \beta_i] \geq \frac{1}{2}(\langle N_A\rangle + \langle N_B\rangle), \quad (6)$$

where the last inequality follows from $\sqrt{\langle\mathbf{J}^2\rangle + \frac{1}{4}} - \frac{1}{2} \geq \frac{1}{2}\langle N\rangle$, which is a direct consequence of the relation $\mathbf{J}^2 = \frac{N}{2}(\frac{N}{2}+1)$. Since $N = N_A + N_B$, this concludes the proof of our criterion. Thus every state that has $\langle\mathbf{J}^2\rangle/\langle N\rangle < \frac{1}{2}$ is entangled. This is a tight bound. There are separable states that reach $\langle\mathbf{J}^2\rangle/\langle N\rangle = \frac{1}{2}$, for example, the product state $|2j\rangle_{a_h}|0\rangle_{a_v}|0\rangle_{b_h}|2j\rangle_{b_v}$, which in spin notation corresponds to $|j_A = j, m_A = j\rangle \otimes |j_B = j, m_B = -j\rangle$.

It should be emphasized that our criterion is sufficient, but not necessary. There are entangled states that are not approximate singlets. Since the Hamiltonian Eq. (1) is quadratic, Eq. (2) is a Gaussian state, such that the criteria of Refs. [10] are both sufficient and necessary. However, their application to our states would require the measurement of quadrature amplitudes of the fields, necessitating homodyne detection. Our criterion is specifically designed for the class of states under consideration and for polarization observables. Other criteria for these observables were derived in Ref. [11] for spin-squeezed states and in Refs. [4,12] for entangled fluctuations around macroscopic beams.

The quantities $\langle\mathbf{J}^2\rangle$ and $\langle N\rangle$ are simple to calculate, such that the effects of various imperfections can be studied with ease. We start by investigating the effect of loss. Loss in a general mode c corresponds to a transformation $c \rightarrow \sqrt{\eta}\,c + \sqrt{1-\eta}\,d$, where d is an empty mode and η is the transmission coefficient. Let us start by assuming that the modes a_h and a_h suffer an equal amount of loss described by η_A, while the b modes have a transmission η_B. This leads to the transformations

$$\langle (\mathbf{J}^{A,B})^2\rangle \rightarrow \eta_{A,B}^2\langle (\mathbf{J}^{A,B})^2\rangle + \tfrac{3}{4}\eta_{A,B}(1-\eta_{A,B})\langle N_{A,B}\rangle, \quad \langle \mathbf{J}^A\cdot\mathbf{J}^B\rangle \rightarrow \eta_A\eta_B\langle \mathbf{J}^A\cdot\mathbf{J}^B\rangle. \tag{7}$$

The state before losses, Eq. (2), has $\langle(\mathbf{J}^A)^2\rangle = \langle(\mathbf{J}^B)^2\rangle = -\langle\mathbf{J}^A\cdot\mathbf{J}^B\rangle$, $\langle N_A^2\rangle = \langle N_B^2\rangle = \langle N_A N_B\rangle$, and $\langle N_A\rangle = \langle N_B\rangle = \langle N\rangle/2$, which leads to the following expression for the total angular momentum after losses:

$$\langle \mathbf{J}^2\rangle \rightarrow (\Delta\eta)^2\langle(\mathbf{J}^A)^2\rangle + \tfrac{3}{8}[\eta_A(1-\eta_A)+\eta_B(1-\eta_B)]\langle N\rangle, \tag{8}$$

where $\Delta\eta = \eta_A - \eta_B$. The first term, which depends on $\Delta\eta$, is of order $\langle N\rangle^2$, while the second term is only $O(\langle N\rangle)$. Equation (8) together with our entanglement criterion implies the condition $\Delta\eta \lesssim 2\sqrt{2}/\sqrt{\langle N\rangle}$. The losses (including detection efficiencies) in the a and b modes thus have to be well balanced in order to observe entanglement for large photon numbers. An equivalent requirement was met for $\langle N\rangle$ of order 10^6 in the experiment of Ref. [13] that demonstrated the strong photon number correlations of pulsed twin beams by direct integrative detection. An analogous condition can be derived for a difference in losses between different polarization modes. If all modes suffer the same amount of loss, described by a transmission η, then only the second term in Eq. (8) remains, leading to a loss-induced correction to the ratio $\langle\mathbf{J}^2\rangle/\langle N\rangle$ of $\frac{3(1-\eta)}{4}$, taking into account that the losses also transform $\langle N\rangle$ into $\eta\langle N\rangle$. This gives a critical transmission value $\eta_c = 1/3$, above which entanglement is provable by our criterion. The entanglement is thus surprisingly robust under balanced losses.

So far we have considered a situation where first the ideal state of Eq. (2) is created, and then it is subjected to loss. However, in the cavity setup of Fig. 1, which is required to achieve high photon numbers, photon creation (in the nonlinear crystal) and loss (in the crystal and all other optical elements) happen effectively simultaneously. It is convenient to transform to a new basis of modes given by $c_1 = (1/\sqrt{2})(a_h + b_v)$, $c_2 = (1/\sqrt{2})(a_h - b_v)$, $c_3 = (1/\sqrt{2})(a_v + b_h)$, $c_4 = (1/\sqrt{2})(a_v - b_h)$. In this basis the Hamiltonian (1) becomes that of four independent, but phase-coherent, squeezers, $H = \frac{i\kappa}{2}[(c_1^\dagger)^2 - (c_2^\dagger)^2 - (c_3^\dagger)^2 + (c_4^\dagger)^2 + \text{H.c.}]$. Introducing the quadrature operators $x_i = (1/\sqrt{2})(c_i + c_i^\dagger)$, $p_i = -(i/\sqrt{2})(c_i - c_i^\dagger)$ gives $H = \frac{\kappa}{2}(x_1p_1 - x_2p_2 - x_3p_3 + x_4p_4) + \text{H.c.}$ Writing down the Heisenberg equations for this Hamiltonian, $\dot{x}_1 = i[H, x_1]$, etc., one sees that $\langle p_1^2\rangle$, $\langle x_2^2\rangle$, $\langle x_3^2\rangle$, and $\langle p_4^2\rangle$ become squeezed exponentially, while the fluctuations in the conjugate quadratures, $\langle x_1^2\rangle$, $\langle p_2^2\rangle$, $\langle p_3^2\rangle$, $\langle x_4^2\rangle$ grow correspondingly. In the presence of losses, the Heisenberg equations have to be replaced by Langevin equations of the form $\dot{x}_1 = \kappa(t)x_1 - \lambda x_1 + f_{x1}(t)$, $\dot{p}_1 = -\kappa(t)p_1 - \lambda p_1 + f_{p1}(t)$, and corresponding equations for the other modes. Here the time dependence of $\kappa(t) = \kappa_0 e^{-\Lambda t}$ takes into account the loss of the pump beam while λ is the loss rate of the down-converted light; $f_{x1}(t)$ and $f_{p1}(t)$ are the quantum noise operators associated with the losses [14], satisfying $\langle f_{x1}(t)f_{x1}(t')\rangle = \langle f_{p1}(t)f_{p1}(t')\rangle = -i\langle f_{x1}(t)f_{p1}(t')\rangle = \lambda\delta(t-t')$. Here we have assumed that the loss rate λ is the same for all four down-conversion modes a_h, a_v, b_h, b_v. We will discuss the case of unbalanced loss rates below.

The equation for x_1 can be integrated explicitly, leading to $x_1(t) = e^{\int_0^t k(t')dt'}x_1(0) + \int_0^t dt' e^{\int_{t'}^t k(t'')dt''} f_{x_1}(t')$, where $k(t) = \kappa(t) - \lambda$ and $\int_{t'}^t \kappa(t'')dt'' = \frac{\kappa_0}{\Lambda}(e^{-\Lambda t'} - e^{-\Lambda t})$. There is a corresponding expression for $p_1(t)$ where the sign of $\kappa(t)$ is flipped. To understand what these results imply for the polarization entanglement, one can express the angular momentum in terms of the quadratures x_i, p_i. One finds $J_z = \frac{1}{2}(x_1x_2 + p_1p_2 - x_3x_4 - p_3p_4)$, $J_x = \frac{1}{2}(x_1x_3 + p_1p_3 + x_2x_4 + p_2p_4)$, and $J_y = \frac{1}{2}(-x_1p_4 + x_4p_1 - x_2p_3 + x_3p_2)$. Introducing the generic notation p for the quadratures p_1, x_2, x_3, and p_4, which are squeezed, and x for x_1, p_2, p_3, and x_4, whose fluctuations grow exponentially, one sees that the J_i have the generic form $x\cdot p$, and one finds $\langle\mathbf{J}^2\rangle = 3(\langle x^2\rangle\langle p^2\rangle - \frac{1}{4})$. The total photon number $N = \frac{1}{2}\sum_i(x_i^2 + p_i^2 - 1)$, leading to $\frac{\langle\mathbf{J}^2\rangle}{\langle N\rangle} = \frac{3}{2}\frac{\langle x^2\rangle\langle p^2\rangle - \frac{1}{4}}{\langle x^2\rangle + \langle p^2\rangle - 1}$.

Figure 2 shows the expected time development of the mean photon number $\langle N\rangle$ and the ratio $\langle\mathbf{J}^2\rangle/\langle N\rangle$ for realistic parameter values. The experimentally achievable value for κ can be estimated by extrapolating existing experimental results [5] to higher pump laser intensities. A value of $\tau = \kappa t = 1$ for a single pass through a 2 mm BBO crystal is realistic with weakly focused pump pulses of a few μJ, which is still below the optical damage threshold. The cavity design of Fig. 1 will have loss rates on the percent level. Figure 2 shows that very high photon numbers can be achieved with just a few round-trips. If balanced losses are the only imperfection, then the entanglement is very strong even for large photon numbers,

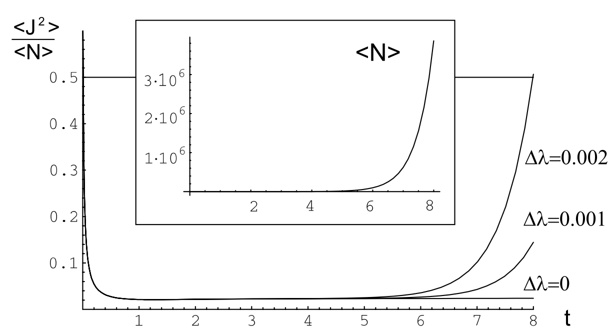

FIG. 2. Time development of the ratio $\langle\mathbf{J}^2\rangle/\langle N\rangle$ and of the mean photon number $\langle N\rangle$. The units are chosen such that $t = 1$ corresponds to a single pass through the crystal. The initial photon creation rate $\kappa_0 = 1$, the mean down-converted photon loss rate $\bar{\lambda} = 0.03$, and the pump loss rate $\Lambda = 0.01$. After eight passes $\langle N\rangle$ reaches the range of millions. The ratio $\langle\mathbf{J}^2\rangle/\langle N\rangle$ is shown for three different values of the loss rate imbalance $\Delta\lambda$, namely, 0, 0.001, and 0.002.

as long as the "laser" is far above threshold, i.e., as long as the rate of creation of entangled photon pairs κ is much larger than the loss rate λ. We are interested in the regime far from saturation (depletion of the pump).

The photon number $\langle N\rangle$ is limited by the requirement of observing entanglement in the presence of other imperfections. In particular, Fig. 2 shows the effect of a difference in the loss rates between the a and b modes. Suppose that the modes a_h and a_v have one loss rate λ_A, while b_h and b_v have a different one λ_B. Then the quadratures x_i, p_i no longer diagonalize the system. For example, x_1 and x_2 satisfy the coupled equations

$$\dot{x}_1 = \kappa(t)x_1 - \bar{\lambda}x_1 - \frac{\Delta\lambda}{2}x_2 + f_{x_1}(t),$$
$$\dot{x}_2 = -\kappa(t)x_2 - \bar{\lambda}x_2 - \frac{\Delta\lambda}{2}x_1 + f_{x_2}(t), \qquad (9)$$

where $\bar{\lambda} = \frac{1}{2}(\lambda_A + \lambda_B)$, $\Delta\lambda = \lambda_A - \lambda_B$, and f_{x_1,x_2} are the appropriate noise operators. There are analogous coupled equations for the pairs p_1 and p_2, x_3 and x_4, and p_3 and p_4. These equations are diagonal for a new basis of modes ξ_i, π_i that is related to the x_i, p_i by a small rotation, which for $\Delta\lambda \ll \kappa$ takes the following simple form: $x_1 = \xi_1 + (\Delta\lambda/4\kappa)\xi_2$, $x_2 = -(\Delta\lambda/4\kappa)\xi_1 + \xi_2$, $x_3 = \xi_3 - (\Delta\lambda/4\kappa)\xi_4$, $x_4 = (\Delta\lambda/4\kappa)\xi_3 + \xi_4$, and identical equations for the p_i in terms of the π_i. In analogy to the case of balanced losses, the quadratures ξ_1, π_2, π_3, and ξ_4 grow exponentially, while the quadratures π_1, ξ_2, ξ_3, and π_4 become squeezed. Because of the small rotation between the old and new diagonal modes, the J_i contain terms that are quadratic in the new large quadratures (ξ_1, π_2, π_3, ξ_4). This leads to an $O(\langle N\rangle^2)$ contribution to $\mathbf{J}^2$. The dominating correction to the ratio $\langle \mathbf{J}^2\rangle/\langle N\rangle$ is $\frac{(\Delta\lambda)^2}{32\kappa^2}\langle N\rangle$, leading to the condition $\Delta\lambda/\kappa \lesssim 4/\sqrt{\langle N\rangle}$ for observing entanglement. Far above threshold (i.e., for $\lambda \ll \kappa$) this is fairly easy to satisfy even for very large photon numbers.

The effects of other imperfections can be studied analogously. The most important one is a phase mismatch between the two twin beams, i.e., a Hamiltonian $H = i\kappa(a_h^\dagger b_v^\dagger - e^{i\phi}a_v^\dagger b_h^\dagger) + \text{H.c.}$ instead of Eq. (1). This gives a correction to the ratio $\langle \mathbf{J}^2\rangle/\langle N\rangle$ whose dominant term is $\frac{1}{16}\phi^2\langle N\rangle$, leading to a condition $\phi \lesssim 4/\sqrt{3}\langle N\rangle$ for observing entanglement. This means that strong entanglement of a million photons can be observed if ϕ is of order $\pi/1000$. This level of precision of optical phases is challenging, but conceivable. Strong entanglement for smaller photon numbers is correspondingly easier to achieve. Another relevant imperfection is a birefringence-related mode mismatch, corresponding to a Hamiltonian $H = i\kappa(a_h^\dagger \tilde{b}_v^\dagger - \tilde{a}_v^\dagger b_h^\dagger) + \text{H.c.}$, where the spatiotemporal modes $\tilde{a}$ and $\tilde{b}$ of the vertical light differ slightly from the modes a and b of the horizontal light. A mode mismatch that affects the a and b modes in a symmetric way leads to a correction to $\langle \mathbf{J}^2\rangle/\langle N\rangle$ that does not grow with $\langle N\rangle$. As before, an asymmetry leads to an $O(\langle N\rangle)$ effect. All significant errors, including the phase mismatch, are related to symmetry breaking between the a and b modes. Geometric symmetry between a and b should be implementable to very high accuracy for the setup of Fig. 1.

In conclusion, the goal of producing strongly entangled singletlike states of very large photon numbers seems realistic with our proposed system. Besides extending the domain where quantum phenomena have been observed, such states would also have interesting applications, for example, in quantum cryptography [8].

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[9] The inequality $|\langle \mathbf{J}\rangle| \leq \sqrt{\langle \mathbf{J}^2\rangle + \frac{1}{4}} - \frac{1}{2}$ can be rewritten more intuitively as $\langle \mathbf{J}^2\rangle \geq |\langle \mathbf{J}\rangle|^2 + |\langle \mathbf{J}\rangle|$. One can always rotate the axes of the coordinate system such that only $\langle J_z\rangle$ is different from zero. The claim is obviously true for a pure state of fixed total spin j. A pure state that is a superposition of components with different j values is effectively a mixed state, because off-diagonal terms between different j do not contribute to $\langle \mathbf{J}\rangle$ and $\langle \mathbf{J}^2\rangle$. For mixed states $\rho = \sum p_i\rho_i$ we have $\sum p_i\langle \mathbf{J}^2\rangle_i \geq \sum p_i[m_i^2 + m_i]$, where we have introduced the notation $m_i = |\langle \mathbf{J}\rangle_i|$. Noting that $|\langle \mathbf{J}\rangle_\rho| = |\sum p_i\langle \mathbf{J}\rangle_i| \leq \sum p_i m_i$ and that $\sum p_i m_i^2 \geq (\sum p_i m_i)^2$ one obtains the desired inequality.
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