Exact Nonstationary Probabilities in the Asymmetric Exclusion Process on a Ring

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The complete solution of the master equation for a system of interacting particles of finite density is presented. By using a new form of the Bethe ansatz, the totally asymmetric exclusion process on a ring is solved for arbitrary initial conditions and time intervals.

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One of the basic problems of nonequilibrium theory of interacting particles is the solving of the master equation

$$\frac{dP(\alpha;t)}{dt} = \sum_{\gamma}[W_{\gamma\alpha}P(\gamma;t) - W_{\alpha\gamma}P(\alpha;t)], \qquad (1)$$

where α, γ are configurations of particles, and $W_{\alpha\gamma}$ is the transition rate from α to γ [1]. Having the solution of Eq. (1), one can get the probability $P(\alpha;t)$ for a system to be in the state α at time t provided it is in the state α_0 at time $t = 0$. However, this program has never been implemented in full not only for realistic many-particle systems but even for simplified one-dimensional models [2]. As a consequence, the problem of exact calculation of general unequal time correlations remains unsolved [3].

Among simple 1D models, the asymmetric exclusion process (ASEP) [4–6] is the most promising candidate for exact solvability of Eq. (1). Indeed, during the last decade, several characteristics of the ASEP have been calculated exactly [3,7–11], mainly for the large time limit. An important step towards a complete nonstationary solution of Eq. (1) for finite time intervals has been done in [12] where the exact probabilities for a finite number of particles on an infinite lattice (i.e., for the zero particle density) have been obtained. Then, a new difficulty has arisen: how to adapt the infinite space dynamics for a closed ring geometry to provide consideration of the physically important case of a finite particle density. Recent interest in this model is due to its close relationship to problems of 1D turbulence, growth of interfaces, one-line traffic, and directed polymers in random media. It is the aim of this Letter to present a complete solution of Eq. (1) for the totally ASEP in a ring containing N sites and P particles.

In continuous time, the totally ASEP is defined by Eq. (1) where transition rates obey the exclusion rule: during any time interval dt, each particle jumps with probability dt to its right if the target site is empty. If the target site is occupied, the move is rejected. The ASEP generates trajectories of P particles starting at points of a ring $0 \le x_1^0 < x_2^0 < \cdots < x_P^0 < L$ and ending at points $0 \le x_1 < x_2 < \cdots < x_P < L$ after an arbitrary number of rotations around the ring during time t. The probability $P(x_1, \ldots, x_P; t|x_1^0, \ldots, x_P^0; 0)$ of finding P particles on the sites $x_1, \ldots, x_P$ at time t with initial occupation $x_1^0, \ldots, x_P^0$ at time $t = 0$ is an integral over all trajectories of P particles allowed by the exclusion rules.

To formulate the main result of this Letter, consider the function [12]

$$F_m(x_i^0, x_j|t) = \sum_{k=0}^{\infty}\binom{k+m-1}{m-1}F_0(x_i^0 - k, x_j|t) \qquad (2)$$

if integer $m > 0$, and

$$F_m(x_i^0, x_j|t) = \sum_{k=0}^{-m}(-1)^k\binom{-m}{k}F_0(x_i^0 - k, x_j|t) \qquad (3)$$

if integer $m < 0$. For $m = 0$,

$$F_0(x_i^0, x_j|t) = B(x_j - x_i^0, t), \qquad (4)$$

where $B(K,t) = e^{-t}t^K/K!$ for $K \ge 0$ and $B(K,t) = 0$ for $K < 0$. Then, for integer $L > 1$ and $0 < P < L$ the probability $P(x_1, \ldots, x_P; t|x_1^0, \ldots, x_P^0; 0)$ is

$$S_P = \sum_{n_1=-\infty}^{\infty} \cdots \sum_{n_P=-\infty}^{\infty} (-1)^{\sum_{i<j}|n_i - n_j|} \det\mathbf{M}. \qquad (5)$$

Elements of the $P \times P$ matrix $\mathbf{M}$ are

$$M_{ij} = F_{s_{ij}}(x_i^0, x_j + n_j L|t), \qquad (6)$$

where

$$s_{ij} = (P-1)n_j - \sum_{k \ne j} n_k + j - i. \qquad (7)$$

The derivation of this result is based on a new form of the Bethe ansatz which retains more details of the picture of interacting trajectories in the ring geometry. A common property of integrable models admitting a two-dimensional space-time representation is that interchanging the end points of two trajectories leads necessarily to their crossing [13]. The idea of the Bethe ansatz is to represent trajectories of interacting particles by a set of free trajectories. Then, using the one-to-one correspondence between intersections and permutations, we can reduce the problem of interacting trajectories to a proper choice of permutations.

The trajectory treatment allows one a qualitative understanding of the structure of the solution. Given the set of starting and ending points $x_i^0, x_i; 1 \le i \le P$ on the infinite lattice, the set of allowed trajectories is defined uniquely. Accounting for the interaction between particles needs $P!$ permutations of the ending points of P trajectories and gives the single determinant expression [12]. On the ring, the set of trajectories depends on the number of loops performed by each particle. As the number of loops is unrestricted, the number of necessary permutations is unrestricted too. This leads to the infinite sum of modified determinants in Eq. (5) taken with appropriate signs.

We start the derivation with the case of two particles $P = 2$. It is convenient to map the trajectories wrapping the ring on an infinite 1D lattice, introducing the coordinates $X_i = x_i + n_i L$, n_i integer, along with the ring coordinates $0 \le x_i < L$. The probability $P(\alpha; t|\alpha_0; 0)$ can be written in the form

$$P(x_1, x_2; t|x_1^0, x_2^0; 0) = \sum_{X_1, X_2} \psi(X_1, X_2; t), \tag{8}$$

where summation is over all X_1, X_2 which satisfy the condition (a) $X_1 = x_1 + n_1^* L$, $X_2 = x_2 + n_2^* L$ or (b) $X_1 = x_2 + n_2^* L$, $X_2 = x_1 + n_1^* L$, n_1^*, n_2^* integer. The function $\psi(X_1, X_2; t)$ is the probability for the first particle to reach the point X_1 from the starting point x_1^0 and for the second particle to reach X_2 from x_2^0 provided the trajectories of particles obey the ASEP rules. The exclusion rule says that the probability $\psi(X_1, X_2; t)$ is nonzero if and only if the trajectories of two particles are compatible, i.e., there is at least one possibility to draw them without colliding. Respectively, $\psi > 0$ if $n_1^* = n_2^*$ for the condition (a) and $n_1^* = n_2^* + 1$ for (b).

Like $P(x_1, x_2; t)$, the function $\psi(X_1, X_2; t)$ satisfies the master equation Eq. (1) which can be converted into the eigenvalue problem

$$\lambda\psi = \psi(X_1 - 1, X_2) + \psi(X_1, X_2 - 1) - 2\psi(X_1, X_2) \tag{9}$$

by the substitution $\psi(X_1, X_2; t) = e^{\lambda t}\psi(X_1, X_2)$ quite similar to the usual one, $P(x_1, x_2; t) = e^{\lambda t}P(x_1, x_2)$. Equation (9) has to be supplemented by boundary conditions

$$\psi(X, X) = \psi(X, X + 1) \tag{10}$$

if the first particle overtakes the second one, and

$$\psi(X, X + L) = \psi(X + 1, X + L) \tag{11}$$

if the second particle, making an additional loop, overtakes the first one.

Instead of the standard Bethe ansatz

$$P(x_1, x_2) = A_{12} z_1^{x_1} z_2^{x_2} + A_{21} z_1^{x_2} z_2^{x_1}, \tag{12}$$

we introduce a "detailed" Bethe ansatz for each pair of compatible coordinates X_1, X_2,

$$\psi(X_1, X_2) = \sum_{n_1, n_2} A_{12}^{(n_1,n_2)} z_1^{x_1 + n_1 L} z_2^{x_2 + n_2 L} + A_{21}^{(n_2,n_1)} z_1^{x_2 + n_2 L} z_2^{x_1 + n_1 L}, \tag{13}$$

where summation is over all n_1, n_2 obeying $n_1 + n_2 = n_1^* + n_2^*$ and parameters A depend on ordering of trajectories. The new form of the Bethe ansatz is a natural extension of Eq. (12). Indeed, the permutation of coordinates x_1, x_2 in Eq. (12) serves for a correction of free trajectories of two particles due to their interaction. The coordinates X_1, X_2 distinguish trajectories with different numbers of rotations n_1, n_2 around the ring and, therefore, the interactions between them need to use all possible permutations starting with the compatible one. The equation $n_1 + n_2 = n_1^* + n_2^*$ is a conservation rule of the total number of rotations under permutations.

From Eq. (9), we have $\lambda = -2 + z_1^{-1} + z_2^{-1}$. The conditions Eqs. (10) and (11) are satisfied if

$$\frac{A_{12}^{(n_1,n_2)}}{A_{21}^{(n_1,n_2)}} = -\frac{(1 - z_1)}{(1 - z_2)} \tag{14}$$

and

$$\frac{A_{12}^{(n_1,n_2)}}{A_{21}^{(n_1-1,n_2+1)}} = -\frac{(1 - z_2)}{(1 - z_1)}. \tag{15}$$

Therefore, the parameters $A_{ij}^{(n_1,n_2)}$ can be found recursively:

$$A_{ij}^{(n_1,n_2)} = (-)^{\pi}(-1)^{|n_1 - n_2|} \frac{f(z_1, z_2)}{(1 - z_1)^{s_{1i}}(1 - z_2)^{s_{2j}}}, \tag{16}$$

where s_{ij} is given by Eq. (7), $f(z_1, z_2)$ is an unknown function, and the sign of permutation $(-)^{\pi}$ is positive for $i, j = 1, 2$ and negative for $i, j = 2, 1$.

Using $z_j = e^{ip_j}$, we can write the general solution

$$\psi(X_1, X_2; t) = \int_0^{2\pi} dp_1 \int_0^{2\pi} dp_2 e^{\lambda t}\psi(X_1, X_2) \tag{17}$$

and determine the function $f(z_1, z_2)$ from the initial conditions. As was noticed in [12] for the case of the infinite lattice, the choice

$$f(z_1, z_2) = e^{-ip_1 x_1^0 - ip_2 x_2^0} \tag{18}$$

and the definition of poles in Eq. (16) by $p_j \to p_j + i0$ provides the correct initial conditions

$$P(x_1, x_2; 0|x_1^0, x_2^0; 0) = \delta_{x_1, x_1^0}\delta_{x_2, x_2^0}. \tag{19}$$

Substituting Eqs. (16) and (18) into Eq. (13) and integrating in Eq. (17), we get

$$\psi(X_1, X_2; t) = \sum_{n_1 + n_2 = n_1^* + n_2^*} (-1)^{|n_1 - n_2|} \det\mathbf{M}, \tag{20}$$

where M is the 2×2 matrix defined by Eq. (6). The summation over all n_1^*, n_2^* leads to Eq. (5) for $P = 2$.

For P particles, the probability $P(\alpha; t|\alpha_0; 0)$ becomes

$$P(x_1, \ldots, x_P; t) = \sum_{\{X\}} \psi(X_1, \ldots, X_P; t), \quad (21)$$

where all X_i satisfy $X_i = x_{\pi(i)} + n^*_{\pi(i)} L$ and π is the permutation of numbers $1, 2, \ldots, P$. The function $\psi(X_1, \ldots, X_P; t)$ is the probability for the ith particle to reach the point X_i from the starting point x_i^0, $i = 1, 2, \ldots, P$ provided the trajectories of the particles obey the ASEP rules. Again, we use the substitution $\psi(X_1, \ldots, X_P; t) = e^{\lambda t} \psi(X_1, \ldots, X_P)$. The eigenvalue problem is a direct generalization of Eq. (9) and has to be supplemented by P boundary conditions corresponding to P possible overtakings:

$$\psi(\ldots, X_i, X_i, \ldots) = \psi(\ldots, X_i, X_i + 1, \ldots) \quad (22)$$

and

$$\psi(X_1, X_2, \ldots, X_1 + L) = \psi(X_1 + 1, X_2, \ldots, X_1 + L). \quad (23)$$

The Bethe ansatz, Eq. (13), takes the form

$$\psi(X_1, \ldots, X_P) = \sum_{\pi} \sum_{\{n\}} A^{(n_{\pi(1)}, \ldots, n_{\pi(P)})}_{\pi(1) \cdots \pi(P)} \prod_{j=1}^{P} z_j^{x_{\pi(j)} + n_{\pi(j)} L}, \quad (24)$$

where the first summation is over all permutations π and the second one obeys $n_1 + \cdots + n_P = n_1^* + \cdots + n_P^*$. The eigenvalue for P particles is

$$\lambda = -P + \sum_{i=1}^{P} \frac{1}{z_i}. \quad (25)$$

To find parameters A from the boundary conditions Eqs. (22) and (23), it is convenient to supply the derivation with a geometrical interpretation. Let us note that two intersecting trajectories are nonequivalent: one of them belongs to the overtaking particle, and we may call it "active." On the contrary, the second particle can be called "passive." In the case of infinite lattice, the active and passive trajectories are ordered: for each pair $i, i + 1$, the trajectory of the ith particle with respect to the $i + 1$ particle is always active. On the ring, each of two trajectories can be active or passive independently of initial conditions. Moreover, one trajectory may intersect another m times if the numbers of rotations differ by m for two particles.

It follows from Eqs. (22) and (23) that

$$\frac{A^{(\ldots, n_i, n_j, \ldots)}_{\ldots ij \ldots}}{A^{(\ldots, n_i, n_j, \ldots)}_{\ldots ji \ldots}} = -\frac{(1 - z_i)}{(1 - z_j)} \quad (26)$$

and

$$\frac{A^{(n_{\pi(1)}, \ldots, n_{\pi(P)})}_{\pi(1) \cdots \pi(P)}}{A^{(n_{\pi(1)} - 1, \ldots, n_{\pi(P)} + 1)}_{\pi(P) \cdots \pi(1)}} = -\frac{(1 - z_{\pi(P)})}{(1 - z_{\pi(1)})}. \quad (27)$$

The ratio of parameters A in Eqs. (26) and (27) is similar to Eqs. (14) and (15) and can be considered as a scattering factor corresponding to the intersection of two particles leading to the permutation of their end points. The trajectories of particles i and $\pi(P)$ in Eqs. (26) and (27) are active, and the trajectories j and $\pi(1)$ passive. We can conclude that each intersection of the active trajectory i gives the factor $(1 - z_i)^{-1}$ in the parameter $A^{(\ldots, n_i, n_j, \ldots)}_{\ldots ji \ldots}$, whereas each intersection of the passive trajectory j gives the factor $(1 - z_j)$. The resulting exponent s_{ij} of $(1 - z_i)$ for the particle starting in x_i^0 and ending in X_j is defined by the difference between total numbers of active and passive intersections, Eq. (7). Summarizing, we can write

$$A^{(n_{\pi(1)}, \ldots, n_{\pi(P)})}_{\pi(1) \cdots \pi(P)} = \frac{(-)^{\pi} (-1)^{\sum_{i<j} |n_i - n_j|} f(z_1, \ldots, z_P)}{(1 - z_1)^{s_{1\pi(1)}} \cdots (1 - z_P)^{s_{P\pi(P)}}}, \quad (28)$$

where the sign of the right-hand side is defined by the number of permutations and $f(z_1, \ldots, z_P)$ is to be found from the initial conditions.

The boundary conditions Eqs. (26) and (27) take into account the pair interactions between particles when two particles are on neighboring sites of the ring. The configurations containing "dense packed" intervals of three and more sites produce no new constraints because the boundary conditions for any pair are independent of coordinates of remaining particles [12].

As above, the choice

$$f(z_1, \ldots, z_P) = \prod_{i=1}^{P} z_i^{-x_i^0} \quad (29)$$

ensures the initial conditions

$$P(x_1, \ldots, x_P; 0 | x_1^0, \ldots, x_P^0; 0) = \delta_{x_1, x_1^0} \cdots \delta_{x_P, x_P^0}. \quad (30)$$

Integrating $e^{\lambda t} \psi(X_1, \ldots, X_P)$ over $p_j = -i \ln z_j$ with parameters A given by Eq. (28), we obtain the function $\psi(X_1, \ldots, X_P; t)$ in the form

$$\psi(X_1, \ldots, X_P; t) = \sum_{\{n\}} (-1)^{\sum_{i<j} |n_i - n_j|} \det \mathbf{M}, \quad (31)$$

where summation $\{n\}$ is restricted by the same conditions as Eq. (24). Continuing the summation in Eq. (21) we obtain the unrestricted sum over all $n_1, \ldots, n_P$ that is Eq. (5).

Finally, we note that the determinant formula Eq. (5) remains valid for the discrete-time ASEP with the parallel update [14]. In this case, we ascribe the probability z to the jump of each particle to an empty site and the probability $y = 1 - z$ to a stay at a given site for the unit time if the jump is not forbidden. Then, the probability $P(x_1, \ldots, x_P; t | x_1^0, \ldots, x_P^0; 0)$ is given by Eqs. (2)–(7) where the Poisson distribution $B(K, t) = e^{-t} t^K / K!$ should be replaced by the Bernoulli probability

$$B(K, t) = \binom{t}{K} z^K y^{t-K}. \tag{32}$$

An advantage of the discrete formulation of the ASEP is a possibility of illustrating obtained results by simple examples. For instance, in the case $P = 2$, $L = 3$, $t = 6$, $x_1^0 = x_1 = 1$, $x_2^0 = x_2 = 2$, we have from Eq. (5) $S_2 = z^{12} + 20z^9y^2 + 30z^9 + z^6y^6 + 30z^6y^5 + 90z^6y^4 + 20z^6y^3 + 20z^3y^6 + 30z^3y^5 + y^6$. All 243 allowed configurations enter into this expression with proper weights.

The solutions Eqs. (2)–(7) of the master equation Eq. (1) give a complete description of the totally ASEP for any initial conditions. Therefore, it may serve as a starting point for evaluating many-particle correlation functions and as a testing ground for more widely applicable approximations.

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