Entanglement Assisted Capacity of the Broadband Lossy Channel

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We calculate the entanglement assisted capacity of a multimode bosonic channel with loss. As long as the efficiency of the channel is above 50%, the superdense coding effect can be used to transmit more bits than those that can be stored in the message sent down the channel. Bounds for the other capacities of the multimode channel are also provided.

Among the zoology of different capacities of quantum channels [1,2], the entanglement assisted classical capacity C_E plays an important role. This quantity has been introduced in [3] to measure the amount of classical information that can be sent through the channel in the presence of an unlimited quantity of prior entanglement between sender and receiver. C_E and its quantum counterpart $Q_E = C_E/2$ (i.e., the amount of qubits that can be sent in the presence of an unlimited quantity of prior entanglement) give upper bounds to the classical and quantum capacities of the channel, including the unassisted capacities whose values are yet to be determined. Moreover, it has been conjectured [4] that the entanglement assisted classical capacity defines a class of equivalences since all channels with the same C_E seem to be able to efficiently simulate one another. Unlike the case of most of the other capacities, it has a closed expression in terms of the quantum mutual information

$$I(\mathcal{N}, \varrho) = S(\varrho) + S(\mathcal{N}[\varrho]) - S((\mathcal{N} \otimes \mathbb{1})[\Phi_\varrho]), \quad (1)$$

where $S(\varrho) = -\text{Tr}[\varrho \log_2 \varrho]$ is the Von Neumann entropy, $\mathcal{N}$ is the map that describes the communication channel, and Φ_ϱ is a purification of the input density matrix ϱ. The value of C_E is the maximum of $I(\mathcal{N}, \varrho)$ over all the possible inputs ϱ to the channel [4,5].

The entanglement assisted capacity for bosonic Gaussian channels was analyzed in [6], where it was shown that the maximization in the expression of C_E can be performed over Gaussian states. These channels are important because they are the basic building blocks of bosonic communication schemes and because they allow one to describe infinite dimensional systems with techniques from finite dimensional linear algebra. In this paper we derive C_E for multimode bosonic channels in the presence of loss and average input energy constraint, and use these results and the techniques developed to provide upper and lower bounds for other channel capacities. We calculate C_E for the multimode channel as the sum of the entanglement assisted capacities of the single

modes maximized over nonsqueezed Gaussian states. In fact, C_E is additive and we show that squeezing the input states does not increase the C_E of a single mode. For generic values of the channel quantum efficiency η we cannot provide an analytical expression for C_E, but we give a general characterization and a numerical solution. For $\eta = 1/2$, the value of C_E can be analytically solved and, interestingly, shown to coincide with the wideband lossless channel capacity [7].

Broadband lossy channel.—In the Heisenberg picture the ith mode of the lossy channel with quantum efficiency η_i evolves as

$$a_i' = \sqrt{\eta_i} \, a_i + \sqrt{1 - \eta_i} \, b_i, \quad (2)$$

where a_i, a_i', and b_i are the annihilation operators of the input, output, and noise modes, respectively. The loss map $\mathcal{N}_i$ for the ith mode arises by tracing away the noise mode b_i (in the vacuum state) and the global loss map $\mathcal{N}$ is the tensor product $\bigotimes_i \mathcal{N}_i$. The channel described by $\mathcal{N}$ maps Gaussian input states into Gaussian output states and is hence a Gaussian channel.

The calculation of C_E for the multimode lossy channel stems from the following three facts: (i) the additivity property of the entanglement assisted capacity, from which the C_E of the channel is calculated as the sum of the C_E of each mode [1,8], i.e.,

$$C_E = \max_{\varrho_j \in \mathcal{H}_j} \left\{ \sum_i I(\mathcal{N}_i, \varrho_i) \right\}, \quad (3)$$

where $\mathcal{H}_j$ is the Hilbert space of the jth mode of the channel, and the max is taken over the states ϱ_i that satisfy the average energy constraint

$$\sum_i \hbar \omega_i N_i = \mathcal{E}, \quad (4)$$

with ω_i the frequency of the ith mode and N_i its average number of photons; (ii) the Holevo-Werner theorem according to which the maximum of $I(\mathcal{N}_i, \varrho_i)$ for Gaussian channels can be evaluated on Gaussian input states [6];

(iii) the fact that squeezing the input does not increase C_E, so that it can be estimated on nonsqueezed inputs: as shown in the appendix, the maximum value of $I(\mathcal{N}_i, \varrho_i)$ (fixing the energy in the ith mode) is obtained when ϱ_i does not contain any squeezing and is given by

$$c_E(N_i, \eta_i) = g(N_i) + g(\eta_i N_i) - g[(1 - \eta_i)N_i], \quad (5)$$

where the function g is defined as

$$g(x) \equiv (x + 1)\log_2(x + 1) - x\log_2(x), \quad (6)$$

for $x \neq 0$ and $g(0) = 0$. The total entanglement assisted capacity is then

$$C_E = \max_{N_j} \sum_i c_E(N_i, \eta_i), \quad (7)$$

where the maximum is taken over the sets $\{N_j\}$ satisfying the energy constraint (4).

The maximization (7) can be performed using the Lagrange multiplier procedure, which, for $\eta \neq 0, 1$, gives the following equation [9]:

$$\left(1 + \frac{1}{N_j}\right)\left(1 + \frac{1}{\eta_j N_j}\right)^{\eta_j} = e^{\omega_j/\Omega}\left(1 + \frac{1}{(1 - \eta_j)N_j}\right)^{1 - \eta_j}, \quad (8)$$

where $1/(\Omega \ln 2)$ is the Lagrange multiplier that must be chosen to satisfy the constraint (4). In general, this equation is difficult to solve analytically, but we can still give some characterization of the solution, at least when all the quantum efficiencies coincide (i.e., $\eta_j = \eta$ for all j). In this case the solution of Eq. (8) is a function of ω_j/Ω and η, i.e., $N_j = \mathcal{F}(\omega_j/\Omega, \eta)$. To derive Ω we use Eq. (4) that becomes

$$\frac{\mathcal{E}}{\hbar} = \sum_i \omega_i \mathcal{F}(\omega_i/\Omega, \eta) \simeq \int_0^\infty \frac{d\omega}{\delta\omega} \omega \mathcal{F}(\omega/\Omega, \eta), \quad (9)$$

where we have replaced the sum over the mode index i with an integral over the mode frequencies, assuming that the minimum frequency interval $\delta\omega$ of the channel is small. With a variable change in the integral (9), we find that $\Omega = \sqrt{2\pi\mathcal{P}/[f(\eta)\hbar]}$ where $\mathcal{P} = \mathcal{E}\delta\omega/(2\pi)$ is the wideband channel input power during the transmission time $\mathcal{T} = 2\pi/\delta\omega$ and

$$f(\eta) \equiv \int_0^\infty dx\, x\mathcal{F}(x, \eta). \quad (10)$$

The value of C_E is then obtained placing the solution of Eq. (8) to evaluate the sum (7), i.e.,

$$C_E \simeq \int_0^\infty \frac{d\omega}{\delta\omega} c_E(\mathcal{F}(\omega/\Omega, \eta), \eta). \quad (11)$$

Performing again a change of integration variables, we finally find

$$C_E = \mathcal{T} \frac{1}{\ln 2}\sqrt{\frac{\pi\mathcal{P}}{3\hbar}} C(\eta), \quad (12)$$

where

$$C(\eta) \equiv \frac{\ln 2}{\pi}\sqrt{\frac{3}{2f(\eta)}}\int_0^\infty dx\, c_E(\mathcal{F}(x, \eta), \eta). \quad (13)$$

Notice that, even without knowing the explicit form of the function $C(\eta)$, Eq. (12) gives the exact dependence on the input power of the entanglement assisted capacity for the channel [10]. In particular, the entanglement assisted capacity per unit time of channel use $R_E \equiv C_E/\mathcal{T}$ is proportional to the rate $R_C = \frac{1}{\ln 2}\sqrt{\pi\mathcal{P}/3\hbar}$ of the wideband noiseless bosonic channel without prior entanglement [7], i.e., $R_E = R_C C(\eta)$.

General properties of C_E.—The form of $C(\eta)$ is not easily determined analytically, but we can still calculate it for some values of η. First of all, for $\eta = 0$ all the $c_E(N_i, \eta)$ are null and $C(0) = 0$: no photons arrive, and no bits are transferred. Interestingly, for $\eta = 1/2$ Eq. (8) can be solved analytically and has solution

$$N_j = \frac{1}{e^{\omega_j/\Omega} - 1}. \quad (14)$$

In this case, $f(1/2) = \pi^2/6$ and $C(1/2) = 1$, and hence the entanglement assisted capacity for the $\eta = 1/2$ wideband channel equals the unassisted capacity of the noiseless wideband channel $\mathcal{T}R_C$ [7]: prior entanglement is sufficient to restore perfect transmission for a 50% lossy channel (this result holds also for the single mode channel—see appendix). The solution can be linearized around $\eta = 1/2$ and the first order Taylor expansion of $C(\eta)$ can be obtained as

$$C(\eta) = \tfrac{3}{2}(\eta - \tfrac{1}{2}) + 1 + \mathcal{O}((\eta - 1/2)^2). \quad (15)$$

The case $\eta = 1$ can be completely solved too, given that the Lagrange equation has the same solution (14) of the case $\eta = 1/2$. Here, since $c_E(N_i, 1) = 2c_E(N_i, 1/2)$, we find $C(1) = 2C(1/2) = 2$: the entanglement assisted capacity for the noiseless channel is twice the unassisted capacity as predicted by the superdense coding effect [11]. In Fig. 1(a) $C(\eta)$ is numerically evaluated and plotted along with the linearization (15). The fact that $C(\eta) > 1$ for $\eta > 1/2$ shows that, even in the presence of noise, prior entanglement allows one to transmit more bits than those actually sent in the channel (i.e., $\mathcal{T}R_C$) thanks again to the superdense coding effect. A similar effect has been shown also for the erasure channel [3,12].

An interesting class of lower bounds that provides a good analytical approximation for C_E can be obtained by considering the set (parametrized by $\zeta > 0$)

$$N_j = \frac{\zeta^2}{e^{\zeta\omega_j/\Omega_0} - 1}, \quad (16)$$

where $\Omega_0 = 6\ln 2 R_C/\pi$. Using Eq. (16), we find the bound

$$C(\eta) \geq \{\Lambda(\zeta^2) + \Lambda(\eta\zeta^2) - \Lambda[(1 - \eta)\zeta^2]\}/[\zeta\Lambda(1)], \quad (17)$$
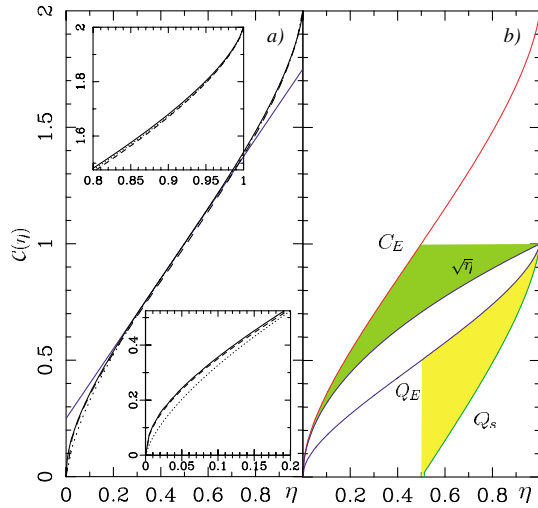
FIG. 1 (color online). (a) Plot, as function of the quantum efficiency η, of the numerical solution for $C(\eta)$ of Eq. (13) (continuous line), of the linearization (15) (gray line), and of the lower bounds (17) with $\zeta = 1$ (dotted line) and $\zeta = 1/\sqrt{\eta}$ (dashed line). The insets show the same graph in the regions of small and large η. The points above $C(\eta) = 1$ (i.e., for $\eta > 1/2$) show where the superdense coding effect allows a lossy channel to beat the capacity of the noiseless channel without prior entanglement. (b) Classical and quantum capacities of the lossy wideband channel. The classical capacity $C/(\mathcal{T}R_C)$ is confined in the dark gray area between the upper bound given by C_E and the lower bound $\sqrt{\eta}$. The quantum capacity $Q/(\mathcal{T}R_C)$ is confined in the light gray area between the upper bound given by the entanglement assisted quantum capacity $Q_E = C_E/2$ and the lower bound Q_s obtained by calculating the coherent information according to Eq. (19). Q is null for $\eta \leq 1/2$.

where $\Lambda(y) \equiv \int_0^\infty dx g(\frac{y}{e^x - 1})$. In particular, the case $\zeta = 1$ [see Fig. 1(a)] corresponds to employing the exact solution for $\eta = 1/2, 1$ of Eq. (14) for any value of η.

Capacity bounds.—The classical capacity C and the quantum capacity Q measure, respectively, the number of bits and qubits that can be sent reliably through the channel per channel use (without the aid of prior entanglement). Unlike the case of C_E, for $\eta \neq 1$ a closed expression for C is not known nor is it known whether this quantity is additive [1]: it may be that entangling successive uses of the channel one can increase the amount of information transmitted. Limiting the analysis to unentangled coding procedures, a lower bound for C can be obtained as [13]

$$C \geq \max_{p_j(\mu), \rho_j(\mu)} \sum_i \mathcal{X}(p_i(\mu), \rho_i(\mu)), \qquad (18)$$

where $\varrho_i = \int d\mu \, p_i(\mu) \rho_i(\mu)$ describes a message in which the "μth letter" $\rho_i(\mu)$ in the ith mode has probability density $p_i(\mu)$ and where $\mathcal{X}$ is the Holevo information $S(\mathcal{N}_i[\varrho_i]) - \int d\mu \, p_i(\mu) S(\mathcal{N}_i[\rho_i(\mu)])$. To estimate the lower bound in Eq. (18), we follow the suggestion of [6] and we evaluate $\mathcal{X}[p_i(\mu), \rho_i(\mu)]$ for

the ith mode using coherent states $\rho_i(\mu) = |\mu\rangle_i \langle\mu|$ weighted with Gaussian probability distribution $p_i(\mu) = \exp[-|\mu|^2/N_i]/(\pi N_i)$, N_i being the average number of photons of the mode. In this case, Eq. (18) becomes $C \geq \max_{N_j} \sum_i g(\eta_i N_i)$, where again the maximum must be taken under the average energy constraint (4). The corresponding Lagrange equation has solution given by Eq. (16) with $\zeta = 1/\sqrt{\eta}$, so that $C \geq \mathcal{T}\sqrt{\eta} R_C$ [14] [see Fig. 1(b)]. Notice that for $\eta = 1$ the equality holds, since the noiseless channel is known to be additive and we reobtain the results of [7]. A closed expression for Q is also not known. However, for $\eta \leq 1/2$ the no-cloning theorem can be used to show that $Q = 0$, as in the case of the erasure channel [12,14]. For $\eta > 1/2$, a lower bound can be obtained evaluating the coherent information $J(\mathcal{N}, \varrho) = S(\mathcal{N}[\varrho]) - S((\mathcal{N} \otimes \mathbb{1})[\Phi_\varrho])$ on unentangled nonsqueezed Gaussian inputs [14,15]. In fact, random quantum codes can send quantum information down a noisy channel at a rate given by the coherent information [16]. In Fig. 1(b) this bound is plotted by solving numerically the corresponding Lagrange equation, which maximizes the expression

$$Q \geq \max_{N_j} \sum_i g(\eta N_i) - g[(1 - \eta)N_i]. \qquad (19)$$

Conclusions.—Up to now only a few realistic channels have been analyzed at the quantum level. In this Letter we studied the wideband bosonic channel with loss, calculating the entanglement assisted capacities C_E and Q_E, and we gave upper and lower bounds on the classical and quantum capacities of this channel. The capacity C_E was shown to scale with the square root of the input power as shown previously for the classical capacities in the noiseless case. Moreover, we saw that the superdense coding effect allows the sender to increase the information transferred above the entropy of the input state if the quantum efficiency is $\eta > 1/2$.

Appendix.—In [6] it has been shown that, for a given value of the correlation matrix α, the quantum mutual information $I(\mathcal{N}, \varrho)$ for a single mode a achieves its maximum value on the Gaussian state

$$\varrho = \frac{\hbar}{2\pi} \int dz \exp[-i(\Delta q, \Delta p) \cdot z^T - z \cdot \alpha \cdot z^T/2], \quad (20)$$

where z is a real bidimensional line vector and q and p the two orthogonal quadratures $q = \sqrt{\hbar/2}(a + a^\dagger)$, $p = -i\sqrt{\hbar/2}(a - a^\dagger)$. In order to evaluate the effect of the squeezing on the quantum mutual information of the single mode channel, it is convenient to introduce the following parametrization for the correlation matrix α:

$$\alpha = \frac{\hbar}{2} \begin{bmatrix} n_0 e^r & c \\ c & n_0 e^{-r} \end{bmatrix}, \qquad (21)$$
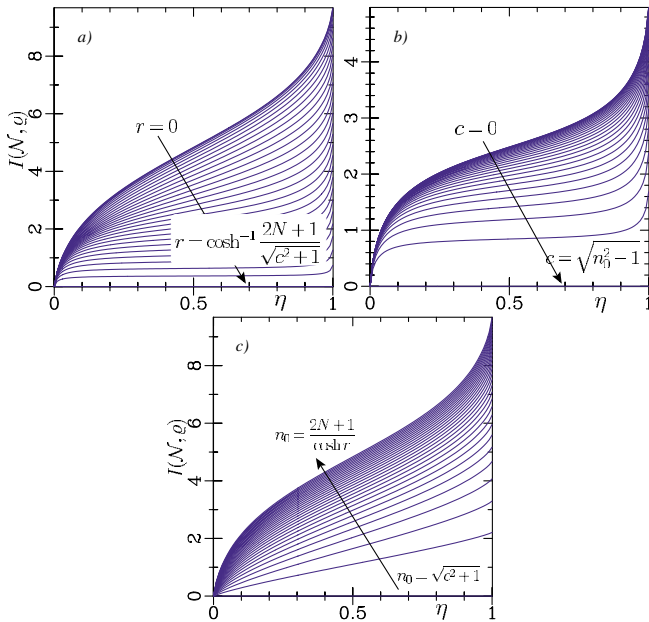
FIG. 2 (color online). Plots of the quantum mutual information $I(\mathcal{N}, \varrho)$ of Eq. (22): (a) $I(\mathcal{N}, \varrho)$ decreases with r (here $c = 0$, $m = 0$); (b) $I(\mathcal{N}, \varrho)$ decreases with c (here $r = 0$, $m = 0$); (c) $I(\mathcal{N}, \varrho)$ increases with n_0, i.e., decreases with m (here $r = 0$, $c = 0$). In all plots $N = 10$.

where r is the squeezing parameter. These parameters are related through the average number of photons N by the conditions $\sqrt{c^2 + 1} \le n_0 = [(2N + 1) - m]/\cosh r$: the first relation derives from the strong version of the uncertainty relation, while the second from the average energy constraint (with $m = \langle q/\hbar \rangle^2 + \langle p/\hbar \rangle^2$). With these definitions the quantum mutual information becomes

$$I(\mathcal{N}, \varrho) = g(\gamma(1)) + g(\gamma(\eta)) - g(\gamma(1 - \eta)), \qquad (22)$$

$$\gamma(\eta) \equiv \sqrt{\left[\eta \lambda_+ + \frac{1 - \eta}{2} \right]\left[\eta \lambda_- + \frac{1 - \eta}{2} \right]} - \frac{1}{2}, \quad (23)$$

where $\lambda_\pm = \frac{1}{2}[n_0 \cosh(r) \pm \sqrt{[n_0 \sinh(r)]^2 + c^2}]$ are the two eigenvalues of $\alpha/\hbar$. Notice that for $\eta = 1$, when all the photons reach the receiver, $I(\mathcal{N}, \varrho)$ is twice the entropy of the initial state, as predicted by the superdense coding effect [11]. In general, one can verify that $I(\mathcal{N}, \varrho)$ is smaller than the initial entropy for $\eta < 1/2$ and greater for $\eta > 1/2$: the effect of superdense coding is, hence, evident only in this last case. Since the eigenvalues $\lambda_\pm$ are related with the average number of photons N as

$$\lambda_+ + \lambda_- = 2N + 1 - m, \qquad (24)$$

one can show that the maximum of $I(\mathcal{N}, \varrho)$ for fixed N is obtained for $\lambda_+ = \lambda_-$. This is equivalent to requiring $r = 0$ [i.e., no energy should be "wasted" in squeezing

the input—see Fig. 2(a)] and $c = 0$ [see Fig. 2(b)]. This last condition attests that the best one can do to convey information is to send maximally mixed states, since the parameter $|c|$ measures the purity of the initial state. Choosing the maximum value of c corresponds to sending a single pure state and conveys no information. Finally, since $I(\mathcal{N}, \varrho)$ is an increasing function of n_0, it can be further maximized by choosing $n_0 = 2N + 1$ [i.e., its maximum allowed value achieved when $\langle q \rangle = \langle p \rangle = 0$—see Fig. 2(c)]. With this choice, Eq. (23) becomes $\gamma_{\text{opt}}(\eta) = \eta N$, which maximizes the quantum mutual information as

$$
\begin{aligned}
c_E(\mathcal{N}, \varrho) &\equiv \max_{\varrho \,|\, \langle a^\dagger a \rangle = N} I(\mathcal{N}, \varrho) \\
&= g(N) + g(\eta N) - g[(1 - \eta)N], \qquad (25)
\end{aligned}
$$

as reported in Eq. (5).

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