Novel Turbulence Trigger for Neoclassical Tearings Mode in Tokamaks

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(Received 13 February 2003; published 24 July 2003)

A stochastic trigger by microturbulence for a neoclassical tearing mode (NTM) is studied. The NTM induces a topological change of magnetic structure and has a subcritical nature. The transition rate of the probability density function for and statistically averaged amplitude of the NTM are obtained. The boundary in the phase diagram is determined as the statistical long time average of the transition conditions. The NTM can be excited by crossing this boundary even in the absence of other global instabilities.

DOI: 10.1103/PhysRevLett.91.045003 PACS numbers: 52.35.Py, 52.35.Ra

Magnetized plasmas are nonuniform and far from thermal equilibrium. Consequently, various kinds of bifurcations can appear [1,2], producing an abrupt change of the topological structure of the magnetic field. In tokamak and in other laboratory plasmas, such a process appears as a magnetohydrodynamic (MHD) instability named the tearing mode [3,4]. It is associated with magnetic field reconnection. Global perturbations with wave numbers perpendicular to the magnetic field are unstable and, due to the plasma resistivity, they can develop radial components that break the field lines. An important problem is now investigated: the possibility of such magnetic surface breaking appearing in ideally stable, low resistivity plasmas.

One possible mechanism is based on a nonlinear instability, the neoclassical tearing mode (NTM) [5–7]. This is a subcritically excited tearing mode under the influence of the pressure gradient. The experiments have shown that such perturbations with a finite amplitude become unstable even for parameters corresponding to linear stability [8–10] and that they can be triggered by other global MHD instabilities (such as the sawtooth) [10,11]. But, in some experiments, the excitation of this instability was produced in the absence of the above trigger [11,12]. The onset conditions of the NTM are not yet clarified, although the suppression of this instability is necessary for stationary operation of high temperature plasma [13]. The rate of stochastic transition was determined at thermal equilibrium [14]. It is expected that in nonequilibrium and turbulent plasmas the transition is triggered by the turbulence but there is no theoretical prediction for the excitation rate of the NTM.

In this Letter, we formulate a Langevin equation for the NTM as a stochastic equation in the presence of the noise source induced by background fluctuations. The statistical properties of NTM amplitude, such as the probability density function (PDF), the rate of excitation, the average of amplitude, the boundary in the phase diagram and its expression, are derived. We show that the stochastic excitation of NTM is possible to occur without seed island if $\beta_p > \beta_{p^*}$ holds. (β_p is the plasma pressure normalized to the poloidal magnetic field pressure.) We note that this mechanism is rather general. For instance, in fluid dynamics it explains the transition of a linearly stable system in a laminar state (flow in a pipe) to a self-sustained turbulent state [15]. The transition is triggered by random disturbances.

The nonlinear instability of the NTM has been discussed, and a dynamical equation of Ohm's law was formulated for the evolution of the amplitude as a deterministic variable

$$\frac{1}{\eta}\frac{\partial}{\partial t}A + \Lambda[A]A = 0, \qquad (1)$$

where $A \equiv \tilde{A}_* q^2 R/B r_s^3 q'$ is the normalized amplitude of the (m, n)-Fourier component of helical vector potential perturbation $\tilde{A}_*$ at the mode rational surface, $r = r_s$, and $-\Lambda$ is the nonlinear growth rate ($-\Lambda > 0$ if unstable). The safety factor $q = rB/B_p R$ as a topological index satisfies the condition $q(r_s) = m/n$ at $r = r_s$. B is the main magnetic field strength, r and R are minor and major radii of torus, $q' = dq/dr$, and m and n are poloidal and toroidal mode numbers, respectively. The time is normalized to poloidal Alfvén transit time, $\tau_{Ap} = qR/\nu_A$ (ν_A is the Alfven velocity) and the length to r_s. The magnetic island width w is expressed as $w = A^{1/2}$. The coefficient η is $\eta = \eta_\parallel \mu_0^{-1} r_s^{-2} \tau_{Ap} = S^{-1}$, where $\eta_\parallel$ stands for a parallel resistivity, and S is the Lundquist number.

An explicit form of the growth rate is given by

$$-\Lambda = 2\Delta' A^{-1/2} - \frac{C_1}{W_1^2 + A^2} + \frac{C_2}{W_2 + A}, \qquad (2)$$

within the neoclassical transport theory, where the first, second, and third terms of the right-hand side (rhs) stand for the effects of current density gradient, polarization drift, and bootstrap current, respectively. The term W_1 represents the cutoff due to the banana orbit effect [16], and we choose a simple model, $W_1 = \rho_b^2 r_s^{-2}$. W_2

represents the cutoff determined by the cross-field energy transport [17]. $C_1 = 2a_{bs}\beta_p \varepsilon^{1/2}\rho_b^2 r_s^{-2} L_q^2 L_p^{-2}$ and $C_2 = 2a_{bs}\beta_p \varepsilon^{1/2} L_q L_p^{-1}$ for the limit of small collisions [7,18,19], ρ_b is the banana width, L_q and L_p are the gradient scale lengths of safety factor and pressure, respectively, ε is the inverse aspect ratio, and a_{bs} is a numerical constant. The parameter Δ' controls the linear stability of the tearing mode [3,4]. When the amplitude A takes finite values, $-\Lambda$ can be positive even if $\Delta' < 0$, because C_1 and C_2 can be positive. Figure 1 illustrates the growth rate as a function of A for the case of $\Delta' < 0$. The marginal stability condition $\Lambda = 0$ can have three solutions at $A \simeq 0$, $A = A_m$, and $A = A_s$ ($A_m < A_s$). A_m and A_s are the threshold and saturation amplitudes, respectively. Near the linear stability boundary, $\Delta' \simeq 0$, they can be estimated as $A_m = C_1 C_2^{-1}$ and $A_s \simeq C_2^2/4\Delta'^2$.

The helical perturbation is subject to a random excitation from the microturbulent noise. The level of noise is evaluated from the Lagrangian nonlinearity terms, and a stochastic equation is obtained instead of Eq. (1):

$$\frac{\partial}{\partial t}A + \eta\Lambda A = s\frac{\delta^2}{a^2}[\phi_h, \Delta A_h]_k - s[\phi_h, A_h]_k - s\frac{\nu_{Te}}{\nu_A}\frac{\delta^2}{a^2}[A_h, \Delta A_h]_k, \quad (3)$$

where $s = aq'/q$ and δ is the collisionless skin depth c/ω_{pe}. ϕ_h is the stream function and A_h is the vector potential of the microscopic turbulence [20]. The suffix h stands for the high mode numbers. The Poisson bracket $[u, v]$ is defined as $(\nabla u \times \nabla v) \cdot \boldsymbol{b}$, and $\boldsymbol{b} = \boldsymbol{B}/B$. $[\cdots]_k$ indicates the Fourier component that matches the test macro mode, and k is the wave number of the macro mode.

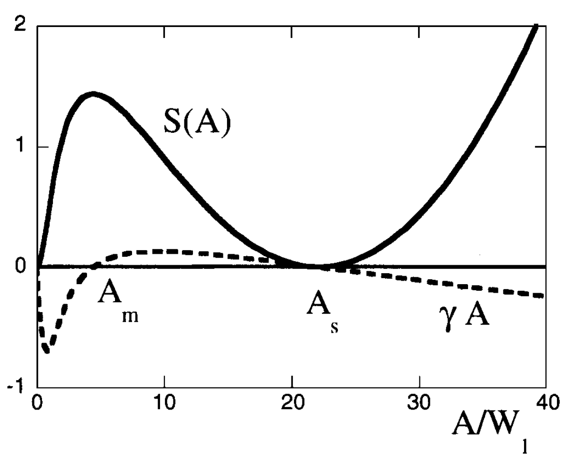

FIG. 1. The normalized growth rate multiplied by amplitude $\gamma A \equiv \Lambda A/C_2$ is shown by the dashed line. Zeros indicate the nonlinear marginal stability conditions for the deterministic model. Normalized nonlinear potential $S(A)/\Gamma C_2 W_1$ is shown by the solid line. (Parameters are $W_1 = W_2$, $C_1/2C_2W_1 = 1$, and $\Delta' W_1^{1/2}/C_2 = -0.0922$.)

We employ a hierarchical approach. The microscopic turbulence is in the nonlinearly marginal state [21] and has much shorter autocorrelation time $\tau_{\rm ac}$ than that of the global perturbation. Fluctuations are statistically independent, and the adiabatic approximation is taken. The saturation levels ϕ_h and A_h can depend on A. We do not consider such dependence here, but it can be introduced in the model.

The rhs of Eq. (3) has two components. One is a coherent part, which has a fixed phase with respect to A. The coherent part would modify η, C_1, and C_2 [19]. We note that the sign of C_1 and C_2 can be changed. The electric induction by microfluctuations has been studied in conjunction with dynamo. The α, β, or γ dynamos have been known [1]. In this Letter, however, we use Eq. (2) as a starting assumption and leave the effects of turbulence on Λ for future studies.

The other is an incoherent part. The relative phase to A changes rapidly in time and contributes to the noise term, being approximated to be random, i.e., $\tilde{S}(t) = gw(t)$, where g is the magnitude and $w(t)$ indicates white noise. $\tilde{S}(t)$ has a quadratic form of ϕ_h and A_h, and the local instantaneous amplitude of $\tilde{S}(t)$ is given as $kk_h^3 C A_h^2$, where numerical constant $C = -sf(\delta^2 r_s^{-2} + k_h^{-2}) + s\sqrt{\beta m_i/m_e}\,\delta^2 r_s^{-2}$ with $f \equiv \phi_h/A_h$ is introduced. Estimations are made as $\Delta A_h = -k_h^2 A_h$ and $|\nabla A_h| = k_h A_h$ for microscopic turbulence, and as $|\nabla A| = kA$ for macro test mode. k_h is the typical mode number of the microfluctuations, the inverse of which is separated from the coherence length of the macro mode. (For a case of ballooning mode turbulence in tokamaks, f is evaluated in Ref. [21] and is of the order unity.) The statistical average $\sqrt{g^2}$ is related to $|\tilde{S}|$ by the law of large numbers. Within the coherent area of global test mode, ℓk^{-1}, a large number ($N = k_h^2 \ell k^{-1}$) of independent kicks contribute to $\tilde{S}(t)$. (ℓ is the radial scale length of the macro mode. N is evaluated by noting a quasi-two-dimensional feature of fluctuations.) The average $\sqrt{g^2}$ is $N^{-1/2}$ times smaller than the instantaneous local value of $|\tilde{S}|$. The magnitude g is evaluated as

$$g^2 = kk_h^{-2}\ell^{-1}|\tilde{S}|^2\tau_{\rm ac} = \ell^{-1}k^3k_h^4C^2A_h^4\tau_{\rm ac}, \quad (4)$$

having a dependence like $g^2 \propto (\tilde{B}_{r,h}/B_\theta)^4\tau_{\rm ac}$. Experimental magnitude is explained later. The stochastic equation of NTM amplitude A is rewritten as

$$\frac{\partial}{\partial t}A + \eta\Lambda A = gw(t), \quad (5)$$

and A is now a stochastic variable. The statistical property of the NTM amplitude A is studied. It is worthwhile to compare it with Kramers's idea for thermal equilibrium [14]. In Eq. (5), there is a nonlinear force but no Einstein drag term common in Brownian theory; the fluctuations from turbulence are decidedly nonthermal unlike standard Langevin theory. The Fokker-Planck equation of

$P(A)$ is deduced from Eq. (5) as

$$\frac{\partial}{\partial \tau}P + \frac{\partial}{\partial A}\left(\eta\Lambda + \frac{1}{2}g\frac{\partial}{\partial A}g\right)P = 0. \quad (6)$$

The stationary solution $P_{eq}(A)$ is expressed as $P_{eq}(A) \propto g^{-1}\exp[-S(A)]$ by use of a nonlinear dissipation function as $S(A) = \int_0^A 2\eta\Lambda(A')g^{-2}A'dA'$ which is proportional to the entropy production rate near the thermal equilibrium [1]. Using Eqs. (2) and (4), one has

$$S(A) = \Gamma\left\{-\frac{4}{3}\Delta' A^{3/2} + \frac{1}{2}C_1\ln\left(1+\frac{A^2}{W_1^2}\right) - C_2\left[A - W_2\ln\left(1+\frac{A}{W_2}\right)\right]\right\}, \quad (7)$$

with $\Gamma = 2S^{-1}\ell k^{-3}k_h^{-4}C^{-2}A_h^{-4}\tau_{ac}^{-1}$. The coefficient Γ shows a characteristic value of the ratio between the dissipation for crossing over the barrier and excitation by turbulence noise. Its magnitude and dependence are discussed at the end of this Letter. The PDF is given as $P_{eq}(A) \propto \exp(\Gamma\frac{4}{3}\Delta' A^{3/2} + \Gamma C_2 A)(1+\frac{A^2}{W_1^2})^{-\Gamma C_1/2} \times (1+\frac{A}{W_2})^{-\Gamma C_2 W_2}$. The PDF has a stretched non-Gaussian exponential form with power-law dependence. The exponential term is determined by the damping by current density gradient and the drive by bootstrap current. The power-law decay is mainly due to the polarization drift effect. The minimum of $S(A)$, i.e., zero of Λ, predicts the peak of PDF and the probable value of A.

For the case of a bistable state, the nonlinear potential $S(A)$ is shown by solid curve in Fig. 1, which has two minima at $A = 0$ and $A = A_s$, separated by a local maximum at $A = A_m$. Statistical transitions take place between these solutions. The dominant (i.e., the most probable) state is determined by the balance between the transition for excitation (from $A = 0$ to $A = A_s$) and the decay (from $A = A_s$ to $A = 0$). The long time average, i.e., the statistical average $\langle A\rangle$, is calculated from the PDF.

Calculating a flux of probability density from Fokker-Planck Eq. (6) [1,22], the frequencies of excitation and decay are expressed as

$$r_{ex} = \frac{\eta\sqrt{\Lambda_0\Lambda_m}}{2\pi}\exp[-S(A_m)], \quad (8a)$$

$$r_{dec} = \frac{\eta\sqrt{\Lambda_s\Lambda_m}}{2\pi}\exp[S(A_s) - S(A_m)], \quad (8b)$$

respectively, where the time rates $\Lambda_{m,s}$ are given as $\Lambda_{m,s} = 2A|\partial\Lambda/\partial A|$ at $A = A_m$ and $A = A_s$. $P_{eq}(A)$ has a peak at $A = 0$. A noise level where the NTM is not excited is evaluated from a local average of A near $A = 0$, being given as $\langle A_0\rangle \sim 0.5(-\Gamma\Delta')^{-2/3}$, and yields $\Lambda_0 = \Lambda(\langle A_0\rangle)$. Note that $\Lambda_{0,m,s}$ are normalized, being of the order unity.

The long time average is given as $\langle A\rangle = (A_s r_{ex} + \langle A_0\rangle r_{dec})(r_{ex} + r_{dec})^{-1}$. $\langle A\rangle$ approaches A_s if $r_{ex} > r_{dec}$ holds. It reduces to $\langle A_0\rangle$ if $r_{ex} < r_{dec}$. The phase boundary for the statistical average is determined by the condition $r_{ex} = r_{dec}$. Apart from a logarithmic dependence, the condition is given by $S(A_s) = 0$. Figure 2 shows the statistical average $\langle A\rangle$, together with threshold and saturation amplitudes (A_m and A_s), as a function of β_p. $\langle A\rangle$ drastically changes across the condition $\beta_p = \beta_{p^*}$, a formula of which is derived as follows. From Eq. (7), the condition $S(A_s) = 0$ is rewritten as $-\frac{4}{3}\Delta' A_s^{3/2} = C_2 A_s - \frac{1}{2}C_1\ln(A_s^2 W_1^{-2})$ where $A_s \gg W_1, W_2$ is assumed. Using the relation $A_s \simeq C_2^2/4\Delta'^2$, we have

$$\Delta' = \Delta'_* \equiv -\sqrt{\frac{1}{12}}C_2^{3/2}C_1^{-1/2}[\ln(3C_1/2C_2W_1)]^{-1/2}, \quad (9)$$

where dominant terms are retained. The boundary Δ'_* is negative and of the order unity. Equation (9) is rewritten as $\Delta' = -\sqrt{\varepsilon L_q/3L_p}\,[\ln(3L_q/2L_p)]^{-1/2}a_{bs}r_s\rho_b^{-1}\beta_p$ by substituting C_1 and C_2. It is reformulated in a form of a critical pressure as $\beta_p = \beta_{p^*} = \sqrt{3L_p/\varepsilon L_q} \times [\ln(3L_q/2L_p)]^{1/2}a_{bs}^{-1}(-\Delta')\rho_b r_s^{-1}$.

An example of the transition frequency is estimated in the following. Near the linear stability condition, $\Delta' \simeq 0$, one has $S(A) = (C_1W_1^{-2} - C_2W_2^{-1})A^2/2 + C_2W_2^{-2}A^3/3 - (C_1W_1^{-4} + C_2W_2^{-3})A^4/4 + \ldots$. The potential barrier $S(A_m)$ is given by the maximum. For the case of $W_2 > W_1$, one has a simple estimate $S(A_m) \simeq (1 - 2C_2W_1^2/C_1W_2)C_1/4 \sim C_1/4$, by keeping the first order correction of W_1/W_2. Substituting it into Eq. (8a), one gets the excitation rate of NTM as

$$r_{ex} \simeq \frac{\eta\sqrt{\Lambda_0\Lambda_m}}{2\pi}\exp\left(-\Gamma\frac{C_1}{4}\right). \quad (10)$$

The parameter Γ is the key for the transition frequency. For L-mode plasmas, when one employs the

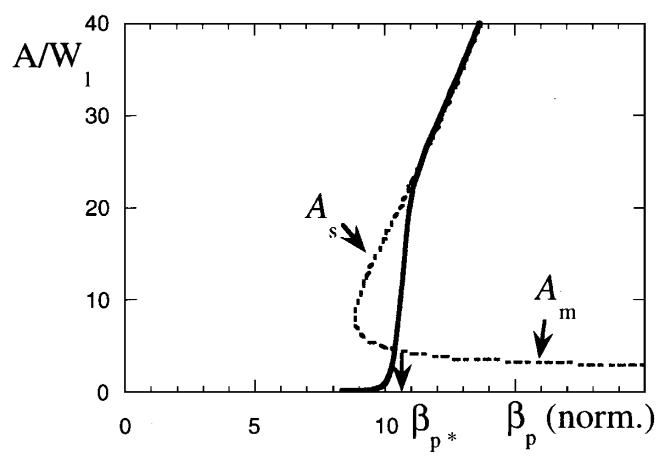

FIG. 2. Amplitude of NTM as a function of the plasma pressure. The solid line shows the statistical average $\langle A\rangle$. A thin dotted line indicates the threshold A_m and saturation amplitude A_s for the deterministic model. Normalized β_p is $C_2/(-\Delta' W_1^{1/2})$, i.e., $[2a_{bs}\varepsilon^{1/2}L_q r_s/\rho_b L_p(-\Delta')]\beta_p$. (Parameters are $W_1 = W_2$, $C_1/2C_2W_1 = 1$, and $\Gamma C_2 W_1 = 5$.)

current-diffusive ballooning mode as micro mode, one has $A_h \simeq 10s\alpha^2(\delta/r_s)^2$, $\phi_h \simeq 10\alpha^{3/2}(\delta/r_s)^2$, and $\tau_{\rm ac} \sim \alpha^{-1/2}$, where $\alpha = -q^2 R d\beta/dr$ is the normalized pressure gradient [23]. Substituting them into the formula of Γ below Eq. (7), one has $\Gamma = 2\ell k^{-3}[-\alpha^{-1/2}(1+\alpha) + s\sqrt{\beta m_i/m_e}]^{-2}10^{-4}s^{-4}\alpha^{-11/2}S^{-1}(\delta/r_s)^{-8}$. The argument $\Gamma C_1/4$ in Eq. (10) may be simplified as $4^{-1}a_{bs}\varepsilon^{1/2}L_q^2 \times L_p^{-2}s^{-2}(m_e/\beta m_i)\ell k^{-3}10^{-4}\alpha^{-11/2}\beta_p S^{-1}\rho_b^2 r_s^6\delta^{-8}$ for $\beta m_i/m_e > 1$. This result shows that when the resistivity becomes so low as to satisfy the condition $S \simeq 10^{-4}(m_e/\beta m_i)\ell k^{-3}\alpha^{-11/2}\rho_b^2 r_s^6\delta^{-8}$, the exponential term becomes of the order of unity, and the transition frequency of the order of η is expected. When the plasma pressure gradient becomes large, a strong turbulence (M mode) has been predicted [21,24]. In this case, $\tilde{A}_h$ is enhanced by the factor of $(\alpha\beta_i m_i/m_e)^{1/2}$. One has $\Gamma C_1/4 \simeq 4^{-1}a_{bs}\varepsilon^{1/2}L_q^2 L_p^{-2}s^{-2}(m_e/\beta m_i)^3\ell k^{-3} \times 10^{-4}\alpha^{-15/2}\beta_p S^{-1}\rho_b^2 r_s^6\delta^{-8}$. The condition of frequent transitions, $\Gamma C_1/4 \sim 1$, is given as $S \simeq 10^{-4}(m_e/\beta m_i)^3\ell k^{-3}\alpha^{-15/2}\rho_b^2 r_s^6\delta^{-8}$. This condition might be easily satisfied in modern tokamaks.

In summary, we have developed a statistical theory for the excitation of nonlinear NTM. The stochastic equation is formulated including the subcritical excitation mechanism of NTM. The rate of transition and statistical average of amplitude are derived, and the phase boundary in plasma parameter space, β_{p^*} or Δ'_*, is obtained. Linearly stable systems are prone to nonlinear instability if $S(A_s) < 0$ holds. The formula is applied to either cases of microfluctuations or of other random MHD activities. The experimental database for the presence of NTM must be compared with the result of the phase boundary derived from the statistical theory. The rate of stochastic transition depends on the microfluctuation level and is evaluated for example cases. However, the boundary is given by $S(A_s) = 0$ and is insensitive to the magnitude of microfluctuations. It is plausible that the stochastic transition without the trigger by large MHD events (e.g., sawtooth or fish-bone instabilities) can be observed in high temperature tokamak plasmas if the condition $\beta_p > \beta_{p^*}$ is satisfied. This explains observations in Refs. [11,12].

Equation (8) is a generalization of the result of thermal equilibrium, i.e., Eq. (476) of Ref. [14] that recovers Arrhenius's law, to the case of the turbulence trigger. Owing to the turbulence trigger, the transition probability is greatly enhanced and the variation of the average $\langle A\rangle$ across $\beta_p = \beta_{p^*}$ becomes less sharp. The energy of microfluctuations is estimated in tokamak turbulence and is about $\delta^2 r_s \lambda_D^{-3}$ times larger than that in thermal equilibrium (§23 of Ref. [1]). In the latter case, Γ is larger by a factor $\delta^4 r_s^2 \lambda_D^{-6}$ and the transition is very difficult to occur.

This Letter provides a theoretical framework for future study. There are a lot of effects and contributions which could be incorporated in the nonlinear statistical theory. These are left for future studies and will give quantitative results.

The authors acknowledge Dr. Y. Miura for useful discussions, as well as Professor M. Vlad and Professor F. Spineanu for improving the manuscript. This work is partly supported by the Grant-in-Aid for Scientific Research of MEXT Japan and by the collaboration programs of NIFS and of RIAM of Kyushu University.

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