Holographic Entropy Bound and Local Quantum Field Theory

Ulvi Yurtsever
Quantum Computing Technologies Group, Jet Propulsion Laboratory, California Institute of Technology, 4800 Oak Grove Drive, Pasadena, California 91109-8099, USA
(Received 6 March 2003; published 23 July 2003)

I show how the holographic entropy bound can be derived from elementary flat-spacetime quantum field theory when the total energy of Fock states is constrained gravitationally. This energy constraint makes the Fock space dimension (whose logarithm is the maximum entropy) finite for both bosons and fermions. Despite the elementary nature of my analysis, it results in an upper limit on entropy in remarkable agreement with the holographic bound, and also provides a microscopic deviation of a more general entropy bound recently introduced by Gour.

DOI: 10.1103/PhysRevLett.91.041302 PACS numbers: 04.70.Dy, 03.70.+k, 04.60.–m, 04.62.+v

An outstanding recent puzzle in gravitational physics is to find a local, microscopic explanation for the "holographic principle" [1–3], which asserts that the maximum entropy that can be stored inside a bounded region $\mathcal{R}$ in 3-space is proportional to the surface area $A(\mathcal{R})$ [as opposed to the volume $V(\mathcal{R})$] of the region:

$$S_{\max}(\mathcal{R}) = \frac{k_B}{4}\frac{A(\mathcal{R})}{l_p^2}, \qquad (1)$$

where k_B is Boltzmann's constant, and $l_p = \sqrt{\hbar G/c^3}$ is the Planck length. The inequality Eq. (1) can only be saturated when the system inside $\mathcal{R}$ is compressed to its Schwarschild radius (the right-hand side being the Bekenstein entropy [4] of the resulting black hole). The holographic relation between $S_{\max}$ and $A(\mathcal{R})$ was first pointed out by 't Hooft [5], and also explored by Bekenstein [6]. The thermodynamic (macroscopic) arguments leading to various similar entropy bounds as well as to the holographic bound Eq. (1) can be found collectively in the review article cited in [2].

The holographic principle presents a puzzle since derivations based on standard (nongravitational) microphysics yield an entropy bound proportional to the volume $V(\mathcal{R})$ instead of the surface area. To discuss this in the simplest microscopic model, let me choose $\mathcal{R}$ to be a standard three-dimensional spacelike cube [7] of size L in Minkowski space, and consider a real, massless (linear) scalar field ϕ confined in $\mathcal{R}$. The Fock space is built out of the modes of the field ϕ, which are the positive-frequency (complexified) solutions of the scalar wave equation $\Box\phi = 0$ that vanish on $\partial\mathcal{R}$. These modes are given (up to normalization) by the solutions $\sin(\vec{k}\cdot\vec{x})e^{-i\omega_{\vec{k}}t}$, where $\omega_{\vec{k}} = c|\vec{k}|$, and the admissible wave vectors $\vec{k}$ are labeled by non-negative integers m_x, m_y, m_z: $(k_x, k_y, k_z) = (\pi/L)(m_x, m_y, m_z)$. I will often use single-letter labels i, j, etc. to denote a composite multi-index such as (m_x, m_y, m_z). Mode counting and summing various quantities over the modes (and all my computations below will be of this kind) can often be simplified via the standard approximation:

$$\sum_{\vec{k}} \to \frac{1}{(\pi/L)^3}\int_{P_+} d^3k = \frac{1}{c^3(\pi/L)^3}\frac{4\pi}{8}\int d\omega\,\omega^2, \qquad (2)$$

where P_+ denotes the "all-positive" octant of $\vec{k}$ space (consisting of positive k_x, k_y, k_z), and the last simplification is available whenever the summed quantity depends only on the mode frequency $\omega = c|\vec{k}|$. Consider, for example, the total number of modes, N. A natural cutoff at or near the Planck frequency, $\omega = 2\pi\mu/\tau_p$, makes N finite, where Planck time $\tau_p = l_p/c$, and μ is a dimensionless constant of order 1 to be specified by a complete theory of the Planck regime (according to naive Planck-scale physics, $\mu = 1$). The total number of modes,

$$N = \sum_i 1 = \frac{L^3}{2\pi^2c^3}\int_0^{2\pi\mu/\tau_p}\omega^2 d\omega = \frac{4\pi\mu^3}{3}\left(\frac{L}{l_p}\right)^3, \qquad (3)$$

is proportional to the volume $V(\mathcal{R}) = L^3$.

The Fock space $\mathcal{H}_F(\mathcal{R})$ for the theory can be constructed as the Hilbert space spanned by orthonormal basis elements of the form

$$|\Psi\rangle = |n_1, n_2, \ldots, n_i, \ldots, n_N\rangle, \qquad n_j \in \mathbb{N}, \qquad (4)$$

which denotes a state with n_i particles occupying mode i. With Fermi statistics, each n_i is restricted to the values $n_i = 0, 1$, while with Bose statistics the n_i can be arbitrarily large integers. The entropy associated to any quantum state of the field is given by $S = -k_B\mathrm{Tr}(\rho\log\rho)$, where ρ is the density matrix of the state. The state with the largest possible entropy is the maximally mixed

$$\rho_{\max} = \frac{1}{\dim\mathcal{H}_F(\mathcal{R})}\mathbf{1}, \qquad (5)$$

the identity operator normalized by the dimension of the Fock space $\mathcal{H}_F(\mathcal{R})$. It follows that maximum entropy is proportional to the log-dimension of $\mathcal{H}_F(\mathcal{R})$:

$$S_{\max} = -k_B\mathrm{Tr}(\rho_{\max}\log\rho_{\max}) = k_B\log\dim\mathcal{H}_F(\mathcal{R}). \qquad (6)$$

The Fock-space dimension (and, hence, the maximum entropy) is infinite for bosons unless the number of

 0031-9007/03/91(4)/041302(4)$20.00

particles in each mode i is constrained by a finite bound. Assuming that the n_i are so constrained,

$$n_i < D, \qquad \forall\ i \tag{7}$$

for some fixed integer D, the number of states of the form Eq. (4) is D^N [$= \dim \mathcal{H}_F(\mathcal{R})$], and Eqs. (6) and (3) give

$$S_{\max} = k_B(\log D)N = \frac{4\pi\mu^3 \log D}{3} k_B \frac{V(\mathcal{R})}{l_p^3}. \tag{8}$$

For fermions (the case $D = 2$), the maximum entropy is proportional to volume. For bosons, we must conclude either that the entropy is unbounded, or we must regularize it with the occupation-number constraints Eq. (7) in which case the bound is again proportional to volume. Even if the constraints D were allowed to depend on the mode frequency ω_i, setting $D_0 \equiv \min\{D_i\}$ ($D_0 \geq 2$), it is clear that $\dim \mathcal{H}_F(\mathcal{R}) \geq D_0^N$, and Eqs. (6) and (3) imply

$$S_{\max} \geq \frac{4\pi\mu^3 \log D_0}{3} k_B \frac{V(\mathcal{R})}{l_p^3}, \tag{9}$$

still in violent disagreement with the holographic bound Eq. (1).

Neglecting the small Casimir-effect contribution to the vacuum stress energy, the regularized total Hamiltonian for the scalar field ϕ can be written in the form $H = \int_{\mathcal{R}} :T_{00}: d^3x = \sum_i \hbar\omega_i a_i^\dagger a_i$, where $a_i^\dagger$, a_i are the usual creation and annihilation operators for the mode i. The total energy of a Fock state of the form Eq. (4) is

$$\langle\Psi|H|\Psi\rangle = \langle\Psi|\sum_i \hbar\omega_i a_i^\dagger a_i|\Psi\rangle = \sum_i \hbar\omega_i n_i. \tag{10}$$

Let me now introduce the ansatz that the Hilbert space of the theory contains only those Fock states $|\Psi\rangle$ for which

$$\langle\Psi|H|\Psi\rangle = \sum_i \hbar\omega_i n_i < E_{\max}, \tag{11}$$

where $E_{\max} \sim (c^4/G)L$ is an upper bound on energy which ensures that the field ϕ is in a stable configuration against gravitational collapse according to semiclassical Einstein equations. More precisely: *the Fock space $\mathcal{H}_F(\mathcal{R})$ of the theory consists of the linear span of the (finitely many) states of the form Eq.* (4) *satisfying the constraint Eq.* (11). It is important to note that this ansatz is consistent with the linear structure of Fock space; any $|\Psi\rangle \in \mathcal{H}_F(\mathcal{R})$ obeys the same energy bound: $\langle\Psi|H|\Psi\rangle < E_{\max}$. For if $|\Psi\rangle$ can be written as a linear combination $|\Psi\rangle = \sum_\alpha c_\alpha|\Psi_\alpha\rangle$, $\sum_\alpha |c_\alpha|^2 = 1$, of the basis states $|\Psi_\alpha\rangle$ satisfying Eq. (11), then, since $|\Psi_\alpha\rangle$ are eigenstates of the Hamiltonian H,

$$\langle\Psi|H|\Psi\rangle = \sum_\alpha |c_\alpha|^2 \langle\Psi_\alpha|H|\Psi_\alpha\rangle < \sum_\alpha |c_\alpha|^2 E_{\max} = E_{\max}.$$

Introducing the dimensionless frequencies Ω_i and the dimensionless energy bound B via

$$\Omega_i \equiv \tau_p\omega_i, \qquad B \equiv \frac{\tau_p}{\hbar} E_{\max}, \tag{12}$$

the ansatz Eq. (11) can be written in the form

$$\sum_i n_i\Omega_i < B. \tag{13}$$

The precise value of B will depend on the details of a self-consistent semiclassical (or fully quantum) theory of gravity; nevertheless, I will assume that it does not differ much from the value predicted by the hoop conjecture [9] applied to the cube $\mathcal{R}$:

$$B = \eta \frac{\sqrt{3}}{4} \frac{L}{l_p}, \tag{14}$$

where η is a dimensionless number of order 1. According to the classical hoop conjecture, $\eta = 1$.

What is the dimension of the Fock space constrained as in Eq. (11)? For both bosons and fermions, the dimension is equal to the combinatorial quantity,

$$\dim\mathcal{H}_F(\mathcal{R}) = W(B) \equiv \text{number of } (n_1, n_2, \ldots, n_N),$$
$$n_i \in \mathbb{N}, \qquad \text{such that } \sum_i n_i\Omega_i < B, \tag{15}$$

the cardinality of the space of solutions to Eq. (13) in non-negative integer N-tuples. With Fermi statistics, the n_i are further constrained by $n_i \in \{0, 1\}$; for bosons, there are no additional constraints. The computation of $S_{\max}$ now reduces to knowing how to count the quantity $W(B)$.

First, the computation for bosons, since Bose statistics clearly leads to the larger dimension: Notice that the inequality Eq. (13) can be written in the form

$$\vec{n} \cdot \vec{\Omega} < B, \tag{16}$$

where the vectors $\vec{n} = (n_1, \ldots, n_N)$ and $\vec{\Omega} = (\Omega_1, \ldots, \Omega_N)$ live in N-dimensional Euclidean space $\mathbb{R}^N$. Geometrically, the quantity $W(B)$ is the number of points of the integer lattice $\mathbb{Z}^N$ which are contained in the convex subset $\mathcal{P}^N \equiv \{\vec{x} \cdot \vec{\Omega} < B, x_i \geq 0\}$ of $\mathbb{R}^N$. $\mathcal{P}^N$ is a polyhedral volume in the positive 2^Nth sector ($x_i \geq 0$) of $\mathbb{R}^N$, bounded by the hyperplane $\{\vec{x} \cdot \vec{\Omega} = B\}$ (see Fig. 1 for the geometry of $\mathcal{P}^N$ for $N = 3$). At first thought, one might be tempted to conclude that $W(B)$ is simply proportional to the volume of $\mathcal{P}^N$, since each unit cell of the integer lattice $\mathbb{Z}^N$ contains on average one lattice point and has unit volume. It is easy to show that the volume of a polyhedron $\mathcal{P}^n$ in $\mathbb{R}^n$ whose vertices (the points where its bounding hyperplane intersects the coordinate axes) are located at $x_i = l_i$, $i = 1, 2, \ldots, n$, is

$$V(\mathcal{P}^n) = \frac{1}{n!} l_1 l_2 \ldots l_n. \tag{17}$$

For $\mathcal{P}^N = \{\vec{x} \cdot \vec{\Omega} < B, x_i \geq 0\}$, these edge lengths l_i are

$$l_i = \frac{B}{\Omega_i}. \tag{18}$$

Using $\Pi_i(B/\Omega_i) = \exp\sum_i \log(B/\Omega_i)$, $V(\mathcal{P}^N)$ can be

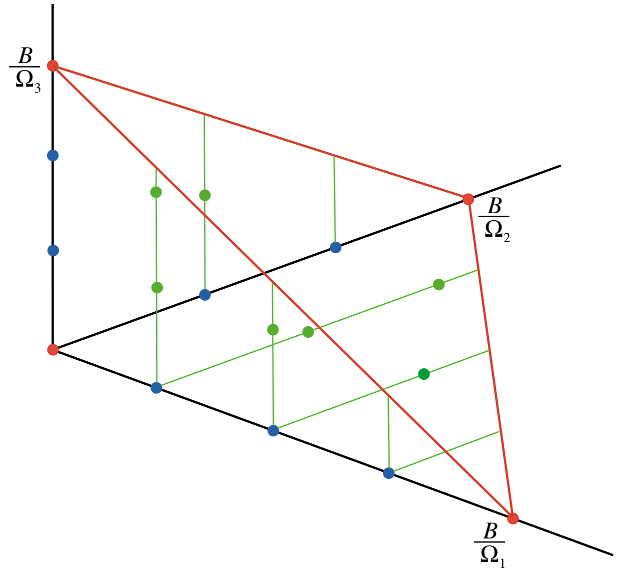

FIG. 1 (color). Example geometry of $\mathcal{P}^3$, with $\lfloor B/\Omega_1 \rfloor = 4$, $\lfloor B/\Omega_2 \rfloor = \lfloor B/\Omega_3 \rfloor = 3$. The boundary of $\mathcal{P}^3$ consists of three $\mathcal{P}^2$ polyhedra (the right-triangular walls), three edges ($\mathcal{P}^1$s), and four vertices. Contributing to $W(B)$ are seven points on the $\mathcal{P}^2$ walls (green dots), seven points on the edges (blue dots), and four vertices (red dots). There is only one contributing interior point (not shown); it is located at $n_1 = n_2 = n_3 = 1$.

calculated with the help of Eq. (2); asymptotically ($L \gg l_p$),

$$V(\mathcal{P}^N) \sim \frac{1}{N!}\exp\left[\frac{4\pi\mu^3}{3}\left(\frac{L}{l_p}\right)^3 \log B\right]. \qquad (19)$$

According to Eqs. (14) and (3) and Stirling's formula $\log N! \sim N\log N - N$, $V(\mathcal{P}^N)$ vanishes exponentially $V(\mathcal{P}^N) \sim \exp[-N\log(N/B)]$ for large L/l_p; $\mathcal{P}^N$ does not even contain a single lattice point of $\mathbb{Z}^N$ in its interior. Solutions of Eq. (13) are distributed skin-deep on the polyhedron $\mathcal{P}^N$; the bulk of the contribution to $W(B)$ comes from points on the boundary of $\mathcal{P}^N$ (Fig. 1). This boundary is comprised of N polyhedra $\mathcal{P}^{N-1}$ of dimension $N-1$, each of which in turn have boundaries made up of $N-1$ $\mathcal{P}^{N-2}$s, and so on. By iterating the reasoning above inductively to the lower-dimensional components of this scaffolding which comprises the boundary of $\mathcal{P}^N$, it is not difficult to show that $W(B)$ can be evaluated as

$$W(B) = 1 + N + \sum_{n=1}^{N}\frac{1}{n!}S_n, \qquad (20)$$

where, for $1 \le n \le N$,

$$S_n \equiv \sum_{i_1<i_2<\cdots<i_n}(l_{i_1}-1)(l_{i_2}-1)\cdots(l_{i_n}-1). \qquad (21)$$

Here I made use of Eq. (17) to compute the *interior* volume of each subpolyhedron $\mathcal{P}^n$ on the boundary. The edge lengths l_{i_k} are reduced by 1 so that only interior points of $\mathcal{P}^n$ contribute to $W(B)$, and overcounting of points that lie on the boundaries of each $\mathcal{P}^n$ is avoided.

Each sum S_n contains $\binom{N}{n}$ summands, resulting in 2^N terms in Eq. (20). How can Eq. (20) be evaluated? The first key observation is a sequence of elementary algebraic identities which leads to a recursion relation for S_n. If I set $S_0 \equiv 1$, and introduce the quantities

$$P_m \equiv \sum_{i=1}^{N}(l_i - 1)^m, \qquad 1 \le m \le N, \qquad (22)$$

then this recursion formula for S_n can be written as

$$S_m = \frac{1}{m}\sum_{j=1}^{m}(-1)^{j-1}P_j S_{m-j}, \qquad 1 \le m \le N. \qquad (23)$$

The next key observation is that in the regime $L/l_p \gg 1$,

$$(P_1)^m \gg P_m, \qquad \forall\ m = 2, \ldots, N. \qquad (24)$$

The proof consists of a straightforward evaluation of the sums P_m via the integral formula Eq. (2), which gives

$$P_1 = \mu^2\left(\frac{L}{l_p}\right)^3 B\left(1 - \frac{4\pi\mu}{3B}\right). \qquad (25)$$

While for higher m (since lowest ω is $\pi c/L$, no true infrared divergences occur at $\omega = 0$), e.g., for $m \ge 4$,

$$P_m \sim \frac{4\pi}{m-3}\left(\frac{BL}{\pi l_p}\right)^m. \qquad (26)$$

Comparison of Eq. (25) with Eq. (26) should make Eq. (24) obvious (see [8] for full details). It follows that in the recursion formula Eq. (23), the first term of the sum dominates over all others, proving that asymptotically

$$S_m \sim \frac{1}{m!}P_1^m, \qquad (27)$$

and, by Eq. (20) and the asymptotic behavior Eq. (19),

$$W(B) = N + q(P_1), \qquad \text{where } q(z) \equiv \sum_{n=0}^{\infty}\frac{z^n}{(n!)^2}. \qquad (28)$$

To discover the entire analytic function $q(z)$, notice that it satisfies the differential equation $q_{,zz} + q_{,z}/z - q/z = 0$, whose solutions are Bessel functions of $\sqrt{z}$. Indeed, $q(z) = I_0(2\sqrt{z})$, the zeroth-order Bessel function of the second kind [10], with asymptotic behavior as $|z| \to \infty$:

$$I_0(z) \sim \frac{e^z}{\sqrt{2\pi z}}\left[1 + \frac{1}{8z} + O\left(\frac{1}{z^2}\right)\right]. \qquad (29)$$

Finally, combining Eqs. (28) and (25),

$$W(B) = N + I_0\left[2\mu\left(\frac{BL^3}{l_p^3}\right)^{1/2}\right], \qquad (30)$$

and Eqs. (14) and (29) gives, asymptotically,

$$S_{\max} = k_B \log W(B) = k_B 3^{1/4}\mu\sqrt{\eta}\left(\frac{L}{l_p}\right)^2, \qquad (31)$$

which [11] is in full agreement with the holographic bound Eq. (1) if $\mu\sqrt{\eta} = 3^{3/4}/2 \cong 1.14$ [note: $A(\mathcal{R}) = 6L^2$].

If the energy bound B is kept general in Eq. (30) instead of substituting from Eq. (14), the asymptotic behavior Eq. (29) leads to the general entropy bound [7]

$$S < S_{\max}(B, L) = k_B 2\mu\left(\frac{BL^3}{l_p^3}\right)^{1/2}, \quad (32)$$

which is precisely the bound recently discovered by Gour [12] via macroscopic thermodynamic arguments [cf. Equation (17) of [12]].

With Fermi statistics, the computation of $W(B)$ involves a different but more straightforward approach, relying on a probabilistic analysis of the distribution of energy over the 2^N subsets (which label the fermionic states) of the set of all modes. The result is

$$S_{\max} = k_B \frac{2}{3\pi\mu} B\left[1 + \log\left(\frac{3\pi\mu}{2}\frac{N}{B}\right)\right], \quad (33)$$

i.e., $S_{\max}$ is proportional to $(L/l_p)\log(L/l_p)$. The full derivation and a discussion of the physical significance of Eq. (33) will be given in [8].

The ansatz Eq. (11) does lead to the correct holographic entropy bound, but how seriously should it be taken? Here are some of the possible consequences of taking Eq. (11) dead seriously as a fundamental physical law.

The commutation relations (CCR) for Bose statistics

$$[a_i, a_j^\dagger] = \delta_{ij} 1 \quad (34)$$

are incompatible with a finite-dimensional Fock space, as can be readily seen by taking the trace of both sides of Eq. (34) [the result is $0 = \delta_{ij} \dim \mathcal{H}_F(\mathcal{R})$]. Indeed, according to the ansatz Eq. (13), whether they obey the Bose CCR or the Fermi CAR, the operators $a_i^\dagger$ must satisfy the algebraic relations

$$\Pi_{i=1}^N (a_i^\dagger)^{n_i} = 0 \qquad \text{whenever} \sum_{i=1}^N n_i \Omega_i \geq B, \quad (35)$$

which imply an algebraic structure drastically different from the CCR (or CAR). One possible way to specify the new algebra (for bosons) is to impose Eq. (35) along with

$$[a_i, a_j^\dagger] = \delta_{ij} 1 + C_{ij}, \quad (36)$$

where C_{ij} are operators which satisfy $\mathrm{Tr} C_{ij} = -\delta_{ij} W(B)$ and whose matrix elements $\langle\Psi|C_{ij}|\Psi'\rangle \approx 0$ for low-energy states $|\Psi\rangle, |\Psi'\rangle$. How can this construction be carried out uniquely, and what are the consequences of the new algebra for physically observable quantities such as expectation values of the stress-energy tensor?

An immediate consequence of Eqs. (35) and (36) is the breakdown of Lorentz invariance at scales much earlier than Planck; namely, at a new temperature scale

$$k_B T_c \sim \frac{\hbar c}{\sqrt{L l_p}}. \quad (37)$$

For a region $\mathcal{R}$ of size $\sim L$, T_c is that temperature at which massless bosons confined in $\mathcal{R}$ have sufficient thermal energy for gravitational collapse [13]. Relative to the characteristic temperature $k_B T \sim \hbar c/L$, T_c corresponds to Lorentz boosts (blueshifts) or order $\gamma \sim z \sim \sqrt{L/l_p}$, whereas the Planck temperature $k_B T_p \sim \hbar c/l_p$ corresponds to (much larger) boosts of order $\gamma \sim z \sim L/l_p$. For feature sizes L at the subnucleon scales, the temperature Eq. (37) is reachable via Lorentz boosts that lie only a few orders of magnitude beyond those envisioned in the large hadron colliders currently under construction [14].

The research described in this paper was carried out at the Jet Propulsion Laboratory, California Institute of Technology, under a contract with the National Aeronautics and Space Administration (NASA), and was supported by grants from NASA and the Defense Advanced Research Projects Agency.

[1] L. Susskind, J. Math. Phys. (N.Y.) **36**, 6377 (1995).

[2] R. Bousso, Classical Quantum Gravity **17**, 997 (2000); Rev. Mod. Phys. **74**, 825 (2002).

[3] E. E. Flanagan, D. Marolf, and R. M. Wald, Phys. Rev. D **62**, 084035 (2000).

[4] J. D. Bekenstein, Phys. Rev. D **7**, 2333 (1973).

[5] G. 't Hooft, in *Salamfestschrift*, edited by A. Ali, J. Ellis, and S. Randjbar-Daemi (World Scientific, Singapore, 1993).

[6] J. D. Bekenstein, Phys. Rev. D **23**, 287 (1981); M. Schiffer, Int. J. Mod. Phys. C **1**, 355 (1990).

[7] My computation depends on the geometry of $\mathcal{R}$ only through the asymptotic distribution of modes as expressed in Eq. (2). The Rayleigh-Jeans-Weyl formula for this asymptotics guarantees that, after the substitution $L^3 \to V(\mathcal{R})$, Eq. (2) remains valid for any sufficiently smooth volume $\mathcal{R}$, generalizing the entire analysis up to and including the derivation of Gour's bound Eq. (32). See [8] for the details.

[8] U. Yurtsever (to be published).

[9] K. S. Thorne, in *Magic without Magic: John Archibald Wheeler*, edited by J. Klauder (Freeman, San Francisco, 1972).

[10] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1973).

[11] The result Eq. (31) was derived for a bosonic massless spin-0 field. For higher-spin massless bosons with two polarization degrees of freedom (e.g., photons), the calculation is exactly the same except the number of modes is doubled along with the quantity P_1, and $S_{\max}$ is larger by a factor of $\sqrt{2}$. Precise agreement with the holographic bound Eq. (1) then requires $\mu\sqrt{\eta} = 3^{3/4}/8 \cong 0.81$.

[12] G. Gour, http://xxx.lanl.gov/abs/gr-qc/0212087.

[13] It is interesting to note that statistical thermodynamics, in which entropy $\propto L^3 T^3$ (Bose or Fermi), gives the "wrong answer" for $S_{\max}$. See [8] for a full discussion.

[14] S. Dimopoulos and G. Landsberg, Phys. Rev. Lett. **87**, 161602 (2001); S. B. Giddings and S. Thomas, http://xxx.lanl.gov/abs/hep-th/0106219.