Fractional Statistics in the Fractional Quantum Hall Effect

Gun Sang Jeon, Kenneth L. Graham, and Jainendra K. Jain
Physics Department, 104 Davey Laboratory, The Pennsylvania State University, University Park, Pennsylvania 16802, USA
(Received 14 March 2003; published 15 July 2003)

A microscopic confirmation of the fractional statistics of the *quasiparticles* in the fractional quantum Hall effect has so far been lacking. We calculate the statistics of the composite-fermion quasiparticles at $\nu = 1/3$ and $\nu = 2/5$ by evaluating the Berry phase for a closed loop encircling another composite-fermion quasiparticle. A careful consideration of subtle perturbations in the trajectory due to the presence of an additional quasiparticle is crucial for obtaining the correct value of the statistics. The conditions for the applicability of the fractional statistics concept are discussed.

DOI: 10.1103/PhysRevLett.91.036801 PACS numbers: 73.43.–f, 71.10.Pm

The fractional statistics concept of Leinaas and Myrheim [1] relies on the property that when particles with infinitely strong short range repulsion are confined in two dimensions, paths with different winding numbers are topologically distinct and cannot be deformed into one another. The particles are said to have statistics θ if a path independent phase $2\pi\theta$ results when one particle goes around another in a complete loop. A half loop is equivalent to an exchange of particles, assuming translational invariance, which produces a phase factor $e^{i\pi\theta} = (-1)^{\theta}$. Nonintegral values of θ imply fractional statistics. There are no fundamental particles in nature that obey fractional statistics. Any fractional statistics objects will have to be emergent collective particles of a nontrivial condensed matter state. Furthermore, they will be necessarily confined to two dimensions: in higher dimensions the notion of a particle going around another is topologically ill defined because any loop can be shrunk to zero without ever crossing another particle.

Even though the explanation of the fractional quantum Hall effect [2] (FQHE) and numerous other remarkable phenomena follows from the composite-fermion (CF) theory with no mention of fractional statistics [3], fractional statistics is believed to be one of the consequences of incompressibility at a fractional filling [4–6], and may possibly be observable in an experiment specifically designed for this purpose. For Laughlin's quasiholes [7] at $\nu = 1/m$, m odd, the statistics was derived explicitly by Arovas, Schrieffer, and Wilczek [4] in a Berry phase calculation, but a similar demonstration of fractional statistics has been lacking at other fractions, or even for the quasiparticles at $\nu = 1/m$. The need for a microscopic confirmation was underscored by Kjønsberg and Myrheim [8] who showed that, with Laughlin's wave function, the quasiparticles at $\nu = 1/m$ do *not* possess well-defined statistics. The reason for the discrepancy remains unclear, but it illustrates that the fractional statistics is rather fragile and cannot be taken for granted.

The objective of this article is to revisit the issue armed with the microscopic composite-fermion theory of the FQHE [9]. A step in that direction has been taken by Kjønsberg and Leinaas [10], whose calculation of the statistics of the "unprojected" CF quasiparticle of $\nu = 1/m$, the wave function for which is different from that of Laughlin's, produced a definite value, the sign of which, however, was inconsistent with general considerations. We confirm below that the statistics is robust to projection into the lowest Landau level (LL), and provide a nontrivial resolution to the sign enigma, which has its origin in very small perturbations in the trajectory due to the insertion of an additional CF quasiparticle. The calculation is extended to $\nu = 2/5$ for further verification of the generality of the concept.

Because the CF theory provides an accurate account of the low energy physics, including incompressibility at certain fractional fillings, it must also contain the physics of fractional statistics, which indeed is the case. The fractional statistics can be derived heuristically in the CF theory as follows [11]. Composite fermions are bound states of electrons and an even number ($2p$) of vortices. When a composite fermion goes around a closed path encircling an area A, the total phase associated with this path is given by

$$\Phi^* = -2\pi(BA/\phi_0 - 2pN_{\rm enc}), \qquad (1)$$

where $N_{\rm enc}$ is the number of composite fermions inside the loop and $\phi_0 = hc/e$ is called the flux quantum. The first term on the right-hand side is the usual Aharonov-Bohm phase for a particle of charge $-e$ going around in a counterclockwise loop. The second term is the contribution from the vortices bound to composite fermions, indicating that each enclosed composite fermion effectively reduces the flux by $2p$ flux quanta. (A note on convention: We will take the magnetic field in the $+z$ direction, the electron charge to be $-e$, and consider the counterclockwise direction for the traversal of trajectories.)

Equation (1) summarizes the origin of the FQHE. The phase in Eq. (1) is interpreted as the Aharonov-Bohm phase from an effective magnetic field: $\Phi^* \equiv -2\pi B^* A/\phi_0$. Replacing N_{enc} by its expectation value $\langle N_{\rm enc}\rangle = \rho A$, where ρ is the two-dimensional density of electrons, we

 0031-9007/03/91(3)/036801(4)$20.00

get

$$B^* = B - 2p\phi_0\rho. \tag{2}$$

The integral quantum Hall effect [12] (IQHE) of composite fermions at CF filling $\nu^* = n$ produces the FQHE of electrons at $\nu = n/(2pn+1)$. At these special filling factors, the effective magnetic field is $B^* = B/(2pn+1)$.

The fractional statistics is also an immediate corollary of Eq. (1). Let us consider the state with CF filling $n < \nu^* < n+1$ and denote by $\eta_\alpha = x_\alpha - iy_\alpha$ the positions where the composite fermions in the top-most partially filled CF level are localized in suitable wave packets. One may imagine a density lump centered at each η_α. An "effective" description in terms of η_α, which will be called CF quasiparticles (CFQPs), can in principle be obtained by integrating out $z_j = x_j - iy_j$. We can conjecture the winding properties of the CFQPs from the underlying CF theory as follows. Consider two CFQPs, sufficiently far from one another that the overlap between them is negligible. According to Eq. (1) the phase a CFQP acquires for a closed loop depends on whether the loop encloses the other CFQP or not. When it does not, the phase is $\Phi^* = -2\pi eB^*A/hc$. The change in the phase due to the presence of the enclosed CFQP is

$$\Delta\Phi^* = 2\pi 2p\Delta\langle N_{\rm enc}\rangle = 2\pi\frac{2p}{2pn+1} \tag{3}$$

because a CFQP has an excess of $1/(2pn+1)$ electrons associated with it relative to the uniform state [producing a local charge of $q^* = -e/(2pn+1)$]. With $\Delta\Phi^* = 2\pi\theta^*$ we get the CFQP statistics parameter

$$\theta^* = \frac{2p}{2pn+1}. \tag{4}$$

This value is consistent, mod 1, with those quoted previously [5,6].

Our goal is to confirm Eq. (4) in a microscopic calculation of the Berry phases. The statistics is given by

$$\theta^* = \oint_C \frac{d\theta}{2\pi}\frac{\langle\Psi^{\eta,\eta'}|i\frac{d}{d\theta}\Psi^{\eta,\eta'}\rangle}{\langle\Psi^{\eta,\eta'}|\Psi^{\eta,\eta'}\rangle} - \oint_C \frac{d\theta}{2\pi}\frac{\langle\Psi^{\eta}|i\frac{d}{d\theta}\Psi^{\eta}\rangle}{\langle\Psi^{\eta}|\Psi^{\eta}\rangle}, \tag{5}$$

where Ψ^η is the wave function containing a single CFQP at η, and $\Psi^{\eta,\eta'}$ has two CFQPs at η and η'. Here we take $\eta = Re^{-i\theta}$, and C refers to the path with R fixed and θ varying from 0 to 2π in the counterclockwise direction. For convenience, we will take $\eta' = 0$.

The calculation of θ^* requires microscopic wave functions which are constructed as follows. The composite-fermion theory maps the problem of interacting electrons at ν into that of weakly interacting composite fermions at ν^*. In order to put these composite fermions at η_α, we first construct the electronic wave function at ν^* with the electrons in the partially filled level at η_α; these are placed in the coherent state wave packets

$$\bar{\phi}^{(n)}_\eta(\vec{r}) = \phi^{(n)}_\eta(\vec{r})\exp[-|z|^2/4l^{*2}], \tag{6}$$

$$\phi^{(n)}_\eta(\vec{r}) = (\bar{z} - \bar{\eta})^n \exp[\bar{\eta}z/2l^{*2} - |\eta|^2/4l^{*2}], \tag{7}$$

where $l = \sqrt{\hbar c/eB}$ and $l^* = (2pn+1)^{1/2}l$ are the magnetic lengths at B and B^*. We then make a mapping into composite fermions *in a manner that preserves distances* (to zeroth order) by multiplying by $\Phi_1^{2p} = \prod_{j<k=1}^N (z_j - z_k)^{2p}\exp[-2p\sum_i |z_i|^2/4l_1^2]$ with $l_1^2 = \hbar c/eB_1 = \hbar c/e\rho\phi_0$, followed by projection into the lowest LL.

To give an explicit example, consider two CFQPs at $\nu = 1/(2p+1)$. The electron wave function at $\nu^* = 1$ with fully occupied lowest LL and two additional electrons in the second LL at η and η' is

$$\Phi_1^{\eta,\eta'} = \begin{vmatrix} \phi^{(1)}_\eta(\vec{r}_1) & \phi^{(1)}_\eta(\vec{r}_2) & \cdot & \cdot & \cdot \\ \phi^{(1)}_{\eta'}(\vec{r}_1) & \phi^{(1)}_{\eta'}(\vec{r}_2) & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot \\ z_1 & z_2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ z_1^{N-3} & z_2^{N-3} & \cdot & \cdot & \cdot \end{vmatrix} e^{-\sum_j |z_j|^2/4l^{*2}}. \tag{8}$$

This leads to the (unnormalized) wave function for two CFQPs at $\nu = 1/(2p+1)$:

$$\Psi^{\eta,\eta'}_{1/(2p+1)} = \mathcal{P}\prod_{i<k=1}^N (z_i - z_k)^{2p} \times \begin{vmatrix} \phi^{(1)}_\eta(\vec{r}_1) & \phi^{(1)}_\eta(\vec{r}_2) & \cdot & \cdot & \cdot \\ \phi^{(1)}_{\eta'}(\vec{r}_1) & \phi^{(1)}_{\eta'}(\vec{r}_2) & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot \\ z_1 & z_2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ z_1^{N-3} & z_2^{N-3} & \cdot & \cdot & \cdot \end{vmatrix} e^{-\sum_j |z_j|^2/4l^2}. \tag{9}$$

Here, $\mathcal{P}$ is the lowest Landau level projection operator, and we have used $l^{*-2} + 2pl_1^{-2} = l^{-2}$, which is equivalent to Eq. (2). Wave functions for one or many CFQPs at arbitrary filling factors can be written similarly. The lowest LL projection can be performed in either one of two ways described in the literature [13]. Our wave functions are similar to those considered in Ref. [10], but not identical.

The integrands in Eq. (5) involve $2N$-dimensional integrals over the CF coordinates, which we evaluate by Monte Carlo method. To determine the O(1) difference between two O(N) quantities on the right-hand side with sufficient accuracy, we use the same importance sampling for both the quantities on the right-hand side, which reduces statistical fluctuations in the difference. The two-CFQP wave function $\Psi^{\eta,\eta'}$ is used as the weight function for both terms in Eq. (5). Approximately 4×10^8

iterations are performed for each point. For $\nu = 1/3$ we have studied systems with $N = 50$, 100, and 200 particles, and the projected wave function is used. In this case, a study of fairly large systems is possible because no explicit evaluation of the determinant is required at each step. For $\nu = 2/5$, it is much more costly to work with the projected wave function, and we have studied only the unprojected wave function for $N = 50$ and 100. The calculation at $\nu = 1/3$ explicitly demonstrates that θ^* is independent of whether the projected or the unprojected wave function is used, or which projection method is used; we assume the same is true at $\nu = 2/5$.

The statistics parameter θ^* is shown in Fig. 1 for $\nu = 1/3$ and $\nu = 2/5$. θ^* takes a well-defined value for large separations. At $\nu = 1/3$ it approaches the asymptotic value of $\theta^* = -2/3$, which is consistent with that obtained in Ref. [10] without lowest LL projection. At $\nu = 2/5$ the system size is smaller and the statistical uncertainty bigger, but the asymptotic value is clearly seen to be $\theta^* = -2/5$. At short separations there are substantial deviations in θ^*; it reaches the asymptotic value only after the two CFQPs are separated by more than $\sim$10 magnetic lengths.

The microscopic value of θ^* obtained above has the same magnitude as θ^* in Eq. (4) *but the opposite sign.* The

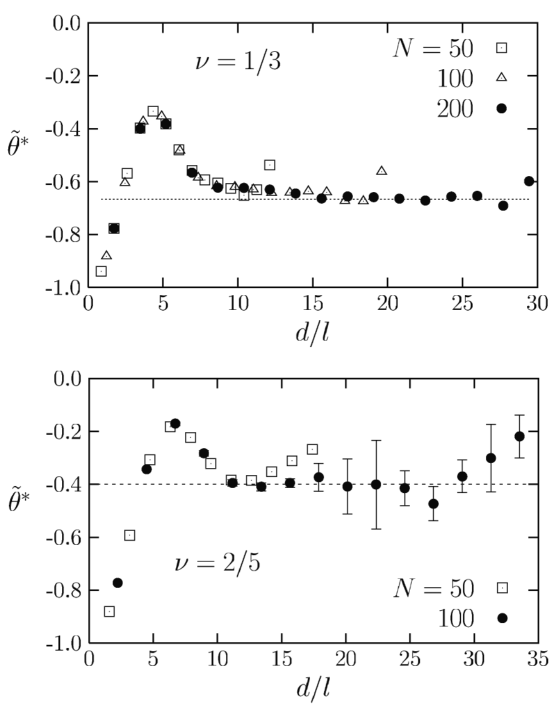

FIG. 1. The statistical angle $\tilde{\theta}^*$ for the CF quasiparticles at $\nu = 1/3$ (upper panel) and $\nu = 2/5$ (lower panel) as a function of $d \equiv |\eta - \eta'|$. N is the total number of composite fermions and l is the magnetic length. (The symbol $\tilde{\theta}^*$ is used rather than θ^* for the statistical angle to remind the reader that the correct interpretation of the results gives $\theta^* = -\tilde{\theta}^*$.) The error bar from Monte Carlo sampling is not shown explicitly when it is smaller than the symbol size. The deviation at the largest d/l for each N is due to proximity to the edge.

sign discrepancy, if real, is profoundly disturbing because it cannot be reconciled with Eq. (1) and would cast doubt on the fundamental interpretation of the CF physics in terms of an effective magnetic field.

To gain insight into the issue, consider two composite fermions in the otherwise empty lowest LL, for which various quantities can be obtained analytically. When there is only one composite fermion at $\eta = Re^{-i\theta}$, it is the same as an electron, with the wave function given by

$$\chi^{\eta} = \exp[\bar{\eta}z/2 - R^2/4 - |z|^2/4]. \tag{10}$$

For a closed loop,

$$\oint_C \frac{d\theta}{2\pi} \frac{\langle \chi^{\eta} | i \frac{d}{d\theta} \chi^{\eta} \rangle}{\langle \chi^{\eta} | \chi^{\eta} \rangle} = -\frac{R^2}{2l^2} = -\frac{\pi R^2 B}{\phi_0}. \tag{11}$$

Two composite fermions, one at η and the other at $\eta' = 0$, are described by the wave function

$$\chi^{\eta,0} = (z_1 - z_2)^{2p}(e^{\bar{\eta}z_1/2} - e^{\bar{\eta}z_2/2})e^{-(R^2+|z_1|^2+|z_2|^2)/4}. \tag{12}$$

Here, we expect $\theta^* = 2p$. However, an explicit evaluation of the Berry phase shows, neglecting $O(R^{-2})$ terms,

$$\oint_C \frac{d\theta}{2\pi} \frac{\langle \chi^{\eta,0} | i \frac{d}{d\theta} \chi^{\eta,0} \rangle}{\langle \chi^{\eta,0} | \chi^{\eta,0} \rangle} = -\frac{R^2}{2l^2} - 2p, \tag{13}$$

which gives $\theta^* = -2p$ for large R. Again, it apparently has the "wrong" sign.

A calculation of the density for $\chi^{\eta,0}$ shows that the actual position of the outer composite fermion is not $R = |\eta|$ but R', given by

$$R'^2/l^2 = R^2/l^2 + 4 \times 2p \tag{14}$$

for large R. This can also be seen in the inset of Fig. 2. The correct interpretation of Eq. (13) is therefore

$$\oint_C \frac{d\theta}{2\pi} \frac{\langle \chi^{\eta,0} | i \frac{d}{d\theta} \chi^{\eta,0} \rangle}{\langle \chi^{\eta,0} | \chi^{\eta,0} \rangle} = -\frac{R'^2}{2l^2} + 2p, \tag{15}$$

which produces $\theta^* = 2p$. The O(1) correction to the area enclosed thus makes a nonvanishing correction to the statistics. (It is noted that the CF quasiparticle at $\eta = 0$ is also a little off center, and executes a tiny circular loop which provides another correction to the phase, but this contribution vanishes in the limit of large R.)

This exercise tells us that an implicit assumption made in the earlier analysis, namely, that the position of the outer CFQP labeled by η remains unperturbed by the insertion of another CFQP, leads to an incorrect value for θ^*. In reality, inserting another CFQP inside the loop pushes the CFQP at η very slightly outward.

To determine the correction at $\nu = n/(2pn + 1)$, we note that the mapping into composite fermions preserves distances to zeroth order, so Eq. (14) should be valid also at $\nu = n/(2pn + 1)$. This is consistent with the shift seen in Fig. 2 for the position of the CFQP. Our earlier result,

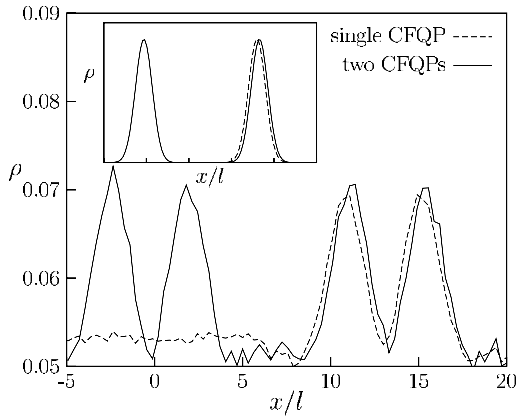

FIG. 2. Density profiles for Ψ^{η} (dashed line) and $\Psi^{\eta,\eta'}$ (solid line) along the x axis at $\nu = 1/3$, with $\eta = 13l$ and $\eta' = 0$. (The uniform state has density $\rho = \nu/2\pi$.) The noise on the curves is a measure of the statistical uncertainty in the Monte Carlo simulation. The CFQP in the second level has a smoke ring shape, with a minimum at its center. The CFQP is located at $x = 13l$ in Ψ^{η}, but is shifted outward in $\Psi^{\eta,\eta'}$. The inset shows the density profiles for χ^{η} (dashed line) and $\chi^{\eta,\eta'}$ (solid line), describing CFQPs in the lowest LL (see the text for definition).

$$\oint_C \frac{d\theta}{2\pi} \frac{\langle \Psi^{\eta,0} | i \frac{d}{d\theta} \Psi^{\eta,0} \rangle}{\langle \Psi^{\eta,0} | \Psi^{\eta,0} \rangle} = -\frac{R^2}{2l^{*2}} - \frac{2p}{2pn+1}, \qquad (16)$$

should be rewritten, using $l^{*2}/l^2 = B/B^* = 2pn + 1$, as

$$\oint_C \frac{d\theta}{2\pi} \frac{\langle \Psi^{\eta,0} | i \frac{d}{d\theta} \Psi^{\eta,0} \rangle}{\langle \Psi^{\eta,0} | \Psi^{\eta,0} \rangle} = -\frac{R'^2}{2l^{*2}} + \frac{2p}{2pn+1}. \qquad (17)$$

When the contribution from the closed path without the other CFQP, $-R'^2/2l^{*2}$, is subtracted out, θ^* of Eq. (4) is obtained. The neglect of the correction in the radius of the loop introduces an error which just happens to be twice the negative of the correct answer.

The fractional statistics of the CFQP should not be confused with the fermionic statistics of composite fermions. The wave functions of composite fermions are single valued and antisymmetric under particle exchange; the fermionic statistics of composite fermions has been firmly established through a variety of facts, including the observation of the Fermi sea of composite fermions, the observation of FQHE at fillings that correspond to the IQHE of composite fermions, and also by the fact that the low energy spectra in exact calculations on finite systems have a one-to-one correspondence with those of weakly interacting fermions [3]. There is no contradiction, however. After all, any fractional statistics in nature *must* arise in a theory of particles that are either fermions or bosons when an *effective* description is sought in terms of certain collective degrees of freedom. The fractional statistics appears in the CF theory when the original particles $\{z\}$ are treated in an average, mean field sense to formulate an effective description in terms of the CFQPs at $\{\eta\}$.

The fractional statistics is equivalent to the existence of an effective locally pure gauge vector potential, with no magnetic field associated with it except at the particle positions [4]. In the present case, the substantial deviation of θ^* from its asymptotic value at separations of up to ten magnetic lengths indicates a core region where the induced vector potential is not pure gauge, thereby imposing a limitation on a model in which the CFQPs are approximated by ideal, pointlike particles with well-defined fractional statistics (anyons). Such an idealization is valid only to the extent that the relevant CFQP trajectories do not involve a significant overlap of CFQPs. Given that there does not exist a strong repulsion between the CFQPs—the inter-CF interaction is very weak and often *attractive* [14]—such trajectories are not precluded energetically, and the anyon model is therefore not a justifiable approximation, except, possibly, for very dilute systems of CFQPs in a narrow filling factor range around $\nu = n/(2pn + 1)$. Any experimental attempt to measure the fractional statistics of the CFQPs must ensure that they remain sufficiently far apart during the measurement process.

Partial support of this research by the National Science Foundation under Grants No. DGE-9987589 (IGERT) and No. DMR-0240458 is gratefully acknowledged. We thank Professor A. S. Goldhaber and Professor J. M. Leinaas for comments.

[1] J. M. Leinaas and J. Myrheim, Nuovo Cimento Soc. Ital. Fis. **37B**, 1 (1977).
[2] D. C. Tsui, H. L. Stormer, and A. C. Gossard, Phys. Rev. Lett. **48**, 1559 (1982)
[3] *Composite Fermions*, edited by O. Heinonen (World Scientific, New York, 1998); *Perspectives in Quantum Hall Effects*, edited by S. Das Sarma and A. Pinczuk (Wiley, New York, 1997); J. K. Jain, Phys. Today **53** No. 4, 39 (2000).
[4] D. Arovas, J. R. Schrieffer, and F. Wilczek, Phys. Rev. Lett. **53**, 722 (1984).
[5] B. I. Halperin, Phys. Rev. Lett. **52**, 1583 (1984).
[6] W. P. Su, Phys. Rev. B **34**, 1031 (1986).
[7] R. B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983).
[8] H. Kjønsberg and J. Myrheim, Int. J. Mod. Phys. A **14**, 537 (1999).
[9] J. K. Jain, Phys. Rev. Lett. **63**, 199 (1989); Phys. Rev. B **40**, 8079 (1989).
[10] H. Kjønsberg and J. M. Leinaas, Nucl. Phys. **B559**, 705 (1999).
[11] A. S. Goldhaber and J. K. Jain, Phys. Lett. A **199**, 267 (1995).
[12] K. v. Klitzing, G. Dorda, and M. Pepper, Phys. Rev. Lett. **45**, 494 (1980).
[13] J. K. Jain and R. K. Kamilla, Int. J. Mod. Phys. B **11**, 2621 (1997).
[14] S.-Y. Lee, V. W. Scarola, and J. K. Jain, Phys. Rev. Lett. **87**, 256803 (2001); A. Wójs and J. J. Quinn, Phys. Rev. B **61**, 2846 (2000).