Chaotic Dynamics of the Fractional Lorenz System

Ilia Grigorenko[1] and Elena Grigorenko[2]

[1]*Institut für Theoretische Physik der Freien Universität Berlin, Arnimallee 14, 14195 Berlin, Germany*

[2]*Irkutsk State Academy of Economics, Lenin street 11, 664015, Irkutsk, Russia*

(Received 8 April 2003; published 17 July 2003)

In this Letter we introduce a generalization of the Lorenz dynamical system using fractional derivatives. Thus, the system can have an effective noninteger dimension Σ defined as a sum of the orders of all involved derivatives. We found that the system with $\Sigma < 3$ can exhibit chaotic behavior. A striking finding is that there is a critical value of the effective dimension Σ_{cr}, under which the system undergoes a transition from chaotic dynamics to regular one.

DOI: 10.1103/PhysRevLett.91.034101 PACS numbers: 05.45.Ac, 02.90.+p

Although fractional derivatives have a long mathematical history, for many years they were not used in physics. One possible explanation of such unpopularity could be that there are multiple nonequivalent definitions of fractional derivatives [1]. Another difficulty is that fractional derivatives have no evident geometrical interpretation because of their nonlocal character [2]. However, during the last ten years fractional calculus starts to attract much more attention of physicists. It was found that various, especially interdisciplinary applications can be elegantly described with the help of the fractional derivatives. As an example, one can mention studies on viscoelastic bodies, polymer physics, phase transitions of fractional order, anomalous diffusion, and a description of the fractional kinetics of chaotic systems (for a review, see [1]). The usefulness of fractional derivatives in quantitative finance [3] and quantum evolution of complex systems [4] was recently demonstrated. One should also mention recent attempts to introduce a local formulation of fractional derivatives [5] and to give some geometrical interpretations [2]. However, most of the studies mentioned above were performed on the basis of *linear* differential equations containing fractional derivatives. The main consequence of this limitation is that the dynamics of such systems cannot be chaotic. According to the Poincaré-Bendixson theorem (see, for example, [6]), chaos cannot occur in two-dimensional systems of autonomous ordinary differential equations. One would have to stress that this theorem is applicable to continuous-time dynamical systems and not to discrete maps. The discrete-time dynamical systems can exhibit chaotic behavior even in one dimension (see, for example, the well-known logistic map [7]). A famous example of a continuous-time three-dimensional system which exhibits chaos is the Lorenz model [8]. The dimension Σ of such system can be defined as a sum of the orders of all involved derivatives. Therefore, by using fractional derivatives of orders $0 < \alpha, \beta, \gamma < 1$ it is possible to obtain a system with an effective noninteger dimension $\Sigma < 3$. A natural question arises whether such a system can exhibit chaotic behavior. In connection with this question one should also mention the work of Hartley *et al.* where the authors studied chaotic motion of the Chua-Hartley system of a fractional order [9]. In this Letter we investigate dynamics of the fractional Lorenz system and we find that can be chaotic with $\Sigma < 3$. We estimate the largest Lyapunov exponent in this case. Moreover, we determine a critical value Σ_{cr}, under which $\Sigma < \Sigma_{cr}$ the dynamics of the considered system becomes regular.

Few definitions of fractional derivatives are known [1]. Probably the best known is the Riemann-Liouville formulation. The Riemann-Liouville derivative of order α and with the lower limit a is defined as

$$\frac{\partial^\alpha}{\partial t^\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \qquad (1)$$

where Γ is the gamma function and n is an integer number chosen in such way that $n - 1 < \alpha < n$. An alternative definition was introduced by Caputo [10]. Caputo's derivative of order α and with the lower limit 0 is a sort of regularization of the Riemann-Liouville derivative and it is defined through

$$\frac{\partial^\alpha}{\partial t^\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau. \qquad (2)$$

The main advantage of the definition Eq. (2) is that Caputo derivative of a constant is equal to zero, that is not the case for the Riemann-Liouville derivative. Substantially, the Caputo fractional derivative is a formal generalization of the integer derivative under the Laplace transformation [3].

Now let us introduce a fractional generalization of the Lorenz system:

$$\frac{\partial^\alpha}{\partial t^\alpha} x = \sigma(y - x), \qquad \frac{\partial^\beta}{\partial t^\beta} y = \rho x - y - xz^r, \qquad \frac{\partial^\gamma}{\partial t^\gamma} z = xy - bz. \qquad (3)$$

Here we assume $0 < \alpha, \beta, \gamma \leq 1$, $r \geq 1$ and the time derivatives are in the Caputo sense. The effective dimension Σ of the system Eq. (3) we define as a sum of the orders

 0031-9007/03/91(3)/034101(4)$20.00

$\alpha + \beta + \gamma = \Sigma$. Throughout this Letter we use the following values of the parameters: $\sigma = 10$, $\rho = 28$, $b = 8/3$, so that in the case $\alpha = \beta = \gamma = r = 1$ the system Eq. (3) reduces to the common Lorenz dynamical system exhibiting chaotic behavior. Generalization of dynamical equations using fractional derivatives could be useful in phenomenological description of viscoelastic liquids such as, for example, human blood [1,11]. The system Eq. (3) is in fact a system of coupled nonlinear integro-differential equations with a weakly singular kernel. This is a computationally expensive problem since for numerical integration it requires $\mathcal{O}(n^2)$ operation counts, where n is the number of sampling points [12].

Let us start from the analytical solution of a linear fractional differential equation:

$$\frac{\partial^\alpha}{\partial t^\alpha} x = Ax + f(t), \qquad x(0) = x_0. \tag{4}$$

With the help of the Laplace transformation [13] one can easily obtain the solution of Eq. (4) in the form:

$$x(t) = x_0 E_\alpha(At^\alpha) + \int_0^t (t-\tau)^{\alpha-1} E_\alpha(A(t-\tau)^\alpha) f(\tau) d\tau, \tag{5}$$

where E_α is the one-parameter Mittag-Leffler function [14] defined by

$$E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + 1)}, \qquad (\alpha > 0). \tag{6}$$

For $\alpha = 1$, E_α reduces to the usual exponent function. The numerical scheme we implemented in our calculations is based on the linearization of the system Eq. (3) on each step of integration and iterative application of Eq. (5). We have checked our numerical scheme by comparison results for Eq. (3), obtained using Eq. (5), and by the standard fourth-order Runge-Kutta method for the case $\alpha = \beta = \gamma = r = 1$.

We integrated Eq. (3) for different values of the parameters α, β, γ, r and different initial conditions. The first finding is that the fractional Lorenz system can exhibit chaotic behavior with the effective dimension $\Sigma < 3$. In Fig. 1 we show the dynamical portrait of the system $\{x(t), y(t), z(t)\}$ using parameters $r = 1$, $\alpha = \beta = \gamma = 0.99$. Thus, the effective dimension of the system is $\Sigma = 2.97 < 3$. We set initial conditions at $t = 0$ as $\{x_0, y_0, z_0\} = \{10, 0, 10\}$. Note that the system exhibits chaotic dynamics similar to the case of the common Lorenz system. Moreover, one probably can also define the set of points which could be characterized as a strange attractor. However, this set is slightly deformed compared to the "classical" Lorenz attractor [15]. We have to stress that it is rather time consuming to define Lyapunov exponents for the nonlocal system, such as Eq. (3). In order to resolve this difficulty we define the largest Lyapunov exponent λ using an implicit procedure developed for the time-series data. With the help of the freeware package TISEAN [16] we estimated the largest Lyapunov exponent λ for the case shown in Fig. 1. We found $\lambda \approx 0.85$ that corresponds to the chaotic regime. Note, that for the common Lorenz system $\lambda \approx 0.906$ [17]. We conclude that decreasing of the effective dimension Σ induces some effective damping in the system. By decreasing the parameters α, β, γ one obtains a further decreasing of the largest Lyapunov exponent. At a certain critical dimension $\Sigma_{\rm cr}$ the dynamics of the system undergoes qualitative changes and becomes regular for any initial condition. This is a new interesting result which, to our knowledge, was not described before. We obtain the lowest value of the system's effective dimension $\Sigma_{\rm cr} \approx 2.91$, for which chaotic regime is still possible. This corresponds to the case $\alpha \approx 0.91$, $\beta = \gamma = 1$. The obtained critical values of the parameters α, β, γ reflect the fact that the first, the linear differential equation in the system Eq. (3) seems to be less "sensitive" to the damping, introduced by a fractional derivative, than other two, nonlinear equations.

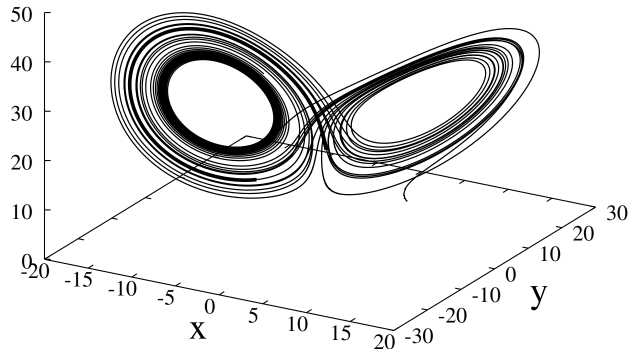

FIG. 1. Dynamical portrait of the fractional Lorenz system using parameters $\alpha = \beta = \gamma = 0.99$ and with the effective dimension $\Sigma = 2.97$. Note formation of the attractor, similar to the Lorenz strange attractor.

If we restrict ourselves for the case of equal derivative orders $\alpha = \beta = \gamma$, the effective critical dimension for this symmetric case is even higher: $\Sigma_{\rm cr}^{\rm sim} \approx 2.94$. In Fig. 2 we show the dynamical portrait of the system setting parameters $r = 1$, $\alpha = \beta = \gamma = 0.97$, with the corresponding effective dimension $\Sigma = 2.91 < \Sigma_{\rm cr}^{\rm sim}$. We use the same initial conditions as for the previous examples. Note, that in this case the system exhibits a strong damping of the oscillations. Dependent on initial conditions, the trajectory of the system is attracted by one of two centers given by

$$(x, y, z) = (\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1). \tag{7}$$

These points can be easily defined from the stationarity condition

$$\frac{\partial^\alpha}{\partial t^\alpha} x = \frac{\partial^\beta}{\partial t^\beta} y = \frac{\partial^\gamma}{\partial t^\gamma} z = 0. \tag{8}$$

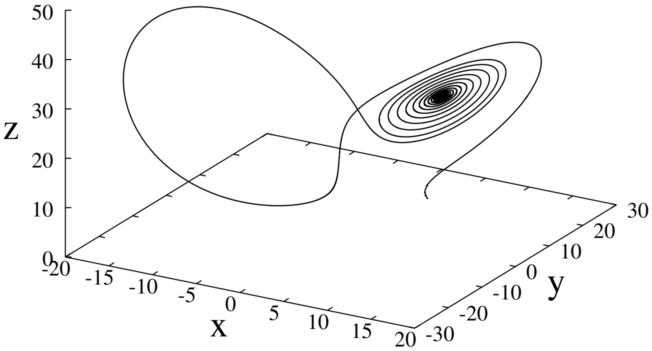

FIG. 2. Dynamical portrait of the fractional Lorenz system using parameters $\alpha = \beta = \gamma = 0.97$ and with the effective dimension $\Sigma = 2.91$. Note, that the strange attractor does not exist and the system is attracted by one of two focuses: $(3\sqrt{8}, 3\sqrt{8}, 26)$ and $(-3\sqrt{8}, 3\sqrt{8}, 26)$.

Note that the stationarity condition Eq. (8) has the usual form because we use the Caputo's fractional derivatives, and it is not applicable, if one uses the Riemann-Liouville formulation Eq. (1).

However, the obtained critical value $\Sigma_{\mathrm{cr}} \approx 2.91$ is not the "universal threshold" for any continuous-time chaotic system of fractional dimension. We found that it is a value that characterizes a particular dynamical system. In order to illustrate this we repeat the simulations shown in the Fig. 2 with the same initial conditions, but with the changed parameter $r = 3$. In Fig. 3 we show the dynamics of the variable $x(t)$. The system Eq. (3) in this case exhibits a "stronger" nonlinearity, which probably compensates the damping effect described above, and one again obtains chaotic behavior.

We also found that under certain conditions the system Eq. (3) can exhibit quasiperiodic oscillations with stable periodic orbits. In Fig. 4 we show an example of such quasiperiodic dynamics of the variable $x(t)$ using parameters values $r = 1$, $\alpha = \beta = 1$, and $\gamma = 0.98$, that corresponds to the effective dimension $\Sigma = 2.98$. We used the same initial conditions as for Figs. 1–3. Note that after some time of transient behavior the system evolves quasiperiodically. For different initial conditions the dynamics of the system shows different limiting cycles having the same two symmetric centers (fixed points) given by Eq. (7).

One can understand the obtained results shown in Figs. 1–4 in the following way. Any chaotic system is characterized by a strong sensitivity to its initial conditions, and the "memory" time of the system can be estimated as $t_{\mathrm{mem}} \approx \lambda^{-1}$, where λ is the largest positive Lyapunov exponent. On the other hand, the introduction of fractional derivatives leads to a nonlocality in time domain [see definition Eq. (2)], which can be interpreted as the presence of long memory. The competition between these two tendencies was the subject of presented investigations.

Now we discuss the question of whether the introduction of fractional derivatives should always lead to stabilization and damping of the chaos in the dynamical system. Let us consider Eqs. (4) and (5) in more detail. In the case of $A < 0$ and in the limit $t \to +\infty$, $E_\alpha(At^\alpha) \propto t^{-\alpha}$ [3]. Therefore, one obtains only a power law convergence of two close trajectories instead of the exponential one for $\alpha = 1$. Thus, even small changes of the orders of derivatives could lead to dramatic changers of the Lyapunov spectrum and of the whole dynamics. One can imagine a situation, when a small decreasing of the order of the derivative α could lead to a stronger sensitivity of the whole nonlinear system to initial conditions. If $A > 0$, in the limit $t \to +\infty$, $E_\alpha(At^\alpha) \propto \exp(A^{1/\alpha} t)$, and

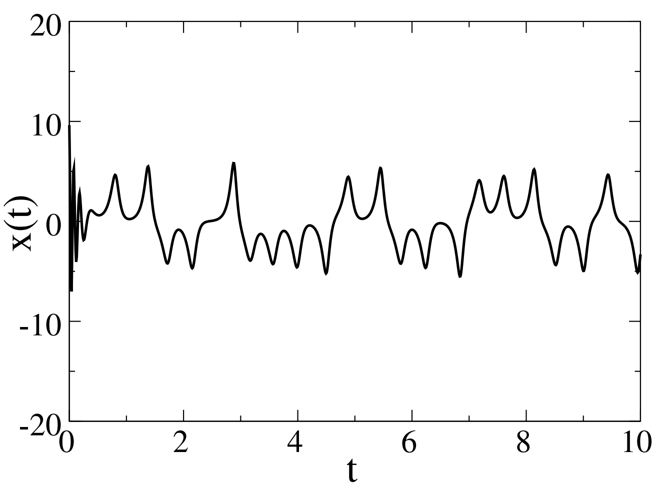

FIG. 3. Evolution of the variable $x(t)$ of the fractional Lorenz system. We use parameters $\alpha = \beta = \gamma = 0.97$, $r = 3$. The effective dimension $\Sigma = 2.91$. Note, that unlike in Fig. 2, the dynamics of the system is chaotic.

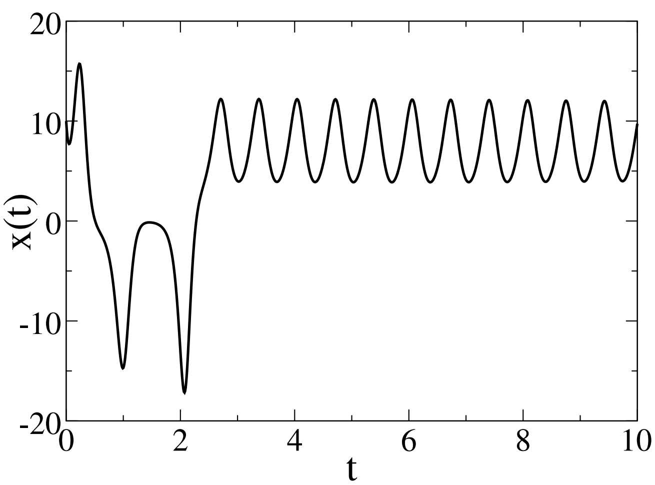

FIG. 4. A quasiperiodic evolution of the variable $x(t)$ of the fractional Lorenz system. We use parameters $\alpha = \beta = 1$, $\gamma = 0.98$, $r = 1$.

one obtains, as in the case of $\alpha = 1$, the exponential divergence of two close trajectories.

In this work we introduced a fractional generalization of the Lorenz model which could be useful in phemenological description of viscoelastic liquids [15]. We study how the dynamics of the system depends on the effective dimension Σ, which now can be a noninteger value. We found that the fractional Lorenz system exhibits rich dynamical properties and can be chaotic with effective dimension Σ less than 3. We also estimated the largest Lyapunov exponent in a particular case. We obtained that the dynamics of the system is strongly sensitive to the values of the orders of the involved derivatives α, β, γ and as the result, to the effective dimension Σ. In general, decreasing of the parameters α, β, γ leads to a damping in the system. We discovered that under the certain critical dimensionality $\Sigma_{\rm cr} \leq 2.91$ chaotic motion of the system is not possible. There are some interesting questions which are still open. Does the lowest universal bound $\Sigma_{\rm cr}^{un}$ exist, under which *any* nonlinear system cannot be chaotic? And how far could it be from the value predicted by the Poincare-Bendixson theorem ($\Sigma_{\rm cr}^{un} = 2$)? Another interesting question is whether the introduction of fractional derivatives of the distributed order [18] in nonlinear systems could help in the description of "edge of chaos," which is characterized by a power law divergence of close trajectories.

[1] *Applications of Fractional Calculus in Physics*, edited by R. Hilfer (World Scientific, River Edge, New Jersey, 2000).
[2] I. Podlubny, math/0110241.
[3] E. Scalas, R. Gorenflo, and F. Mainardi, Physica (Amsterdam) **284A**, 376 (2000); F. Mainardi, M. Raberto, R. Gorenflo, and E. Scalas, Physica (Amsterdam) **287A**, 468 (2000); N. Laskin, Physica (Amsterdam) **287A**, 482 (2000).
[4] D. Kusnezov, A. Bulgac, and G. D. Dang, Phys. Rev. Lett. **82**, 1136 (1999).
[5] K. M. Kolwankar and A. D. Gangal, Phys. Rev. Lett. **80**, 214 (1998).
[6] M.W. Hirsch and S. Smale, *Differential Equations, Dynamical Systems and Linear Algebra* (Academic Press, New York 1965).
[7] E. Ott, *Chaos in Dynamical Systems* (Cambridge University Press, Cambridge, 1994).
[8] E. N. Lorenz, J. Atmos. Sci. **20**, 130 (1963).
[9] T. T. Hartley, C. F. Lorenzo, and H. K. Qammer, IEEE Trans. Circuits Syst. I **42**, 485 (1995).
[10] M. Caputo, Geophys. J. R. Astron. Soc. **13**, 529 (1967).
[11] G. B. Thurston, Biophys. J. **12**, 1205 (1972).
[12] K. Diethelm, Elect. Trans. Num. Anal. **5**, 1 (1997).
[13] I. Podlubny, funct-an/9710005; *Fractional Calculus and Its Applications* (Academic Press, San Diego, 1999).
[14] A. Edérlyi, *Higher Transcendental Functions* (McGraw-Hill, New York, 1955).
[15] I. Grigorenko (to be published).
[16] For the overview of the TISEAN package, see, for example, chao-dyn/9810005.
[17] C. Sparrow, *The Lorenz Equations: Bifurcations, Chaos, and Strange Attractors* (Springer-Verlag, New York, 1982).
[18] A.V. Chechkin, R. Gorenflo, and I. M. Sokolov, Phys. Rev. E **66**, 046129 (2002).