

QCD Inequalities for the Nucleon Mass and the Free Energy of Baryonic Matter

Thomas D. Cohen*

Department of Physics, University of Maryland, College Park, Maryland 20742-4111, USA

(Received 3 April 2003; published 18 July 2003)

The positivity of the integrand of certain Euclidean space functional integrals for two flavor QCD with degenerate quark masses implies that the free energy per unit volume for QCD with a baryon chemical potential μ_B (and zero isospin chemical potential) is greater than the free energy with an isospin chemical potential $\mu_I = (2\mu_B/N_c)$ (and zero baryon chemical potential). The same result applies to QCD with any number of heavy flavors in addition to the two light flavors so long as the chemical potential is understood as applying to the light quark contributions to the baryon number. This relation implies a bound on the nucleon mass: there exists a particle X in QCD (presumably the pion) such that $M_N \geq (N_c m_X/2 I_X)$ where m_X is the mass of the particle and I_X is its isospin.

DOI: 10.1103/PhysRevLett.91.032002

PACS numbers: 12.38.Lg, 14.20.Dh, 21.65.+f

Quantum chromodynamics (QCD) is the theory underlying strong interactions. The theory is not analytically tractable via perturbative and other weak coupling methods except for a limited set of observables in a limited kinematic regime. While certain aspects of the nonperturbative regime of QCD have been explored via lattice QCD [1], it is important to establish as many properties of QCD as possible via rigorous analytic means. QCD inequalities represent an ideal method to do this. An early variant of the approach was based on a demonstration by Nussinov [2] that bounds could be placed on hadronic quantities in a wide class of models inspired by QCD. The approach was greatly strengthened by the realization of Weingarten [3] and Witten [4] that similar bounds could be obtained directly from QCD itself through inequalities based on Euclidean space functional integral representations of physical quantities. Thus, the approach is a method to deduce certain qualitative features of QCD from first principles. While the resulting inequalities have not been proved as theorems at the level of rigor demanded by mathematicians, they make use of only the most standard assumptions made by physicists. For example, one assumes that the theory exists, that physical quantities may be represented via functional integrals, that a Wick rotation from Minkowski space to Euclidean space is innocuous and so on. The field of QCD inequalities is now two decades old; the state of the art is presented in a recent comprehensive review by Nussinov and Lampert [5]. Although the results of QCD inequalities are qualitative, they can serve to supplement understanding gleaned from lattice QCD studies; they provide an analytic means of understanding some of the results of QCD which are both seen in nature and which emerge from numerical studies on the lattice. QCD inequalities may also provide insight into certain properties of QCD which are not tractable on the lattice using Monte Carlo algorithms.

The underlying idea of QCD inequalities is quite simple. One relates a physical quantity of interest to a Euclidean space functional integral over gauge field con-

figurations. If a second quantity can be represented via a functional integral for which the integrand is greater than or equal to the integrand of the initial quantity for all gauge configurations, then one can conclude that the second functional integral is bigger than the first and this in turn allows one to bound one physical quantity by another.

This Letter focuses on the use of QCD inequalities in two apparently unrelated problems in strong interaction physics. The first is bounding the mass of the nucleon from below in terms of other physical observables. This is an old problem. Nussinov derived a bound that the nucleon mass must be greater than or equal to $3m_\pi/2$ in the context of QCD-inspired models. Weingarten, in his original paper, attempted to bound the mass of the nucleon as some multiple of the pion mass directly from QCD [3]. Unfortunately, the only rigorous method found required the study of QCD with six or more degenerate light flavors. An alternative approach was also suggested by Weingarten; however, it used plausible but not provable assumptions about bounds on the quark propagator in the presence of an arbitrary gauge potential. Subsequently, Nussinov and Sathiapalan [6] showed that in the large N_c limit of QCD with two degenerate flavors, the nucleon mass is bounded by $M_N > (N_c m_\pi/2)$. However, previously no rigorous lower bound has been obtained for the nucleon mass directly from QCD at finite N_c for two degenerate flavors. Such a bound is derived here.

The second issue discussed in this Letter is the problem of finding a lower bound on the free energy of QCD at nonzero baryon chemical potential, or more precisely a baryon chemical potential associated with the two light degenerate flavors. This second problem is significant because for the zero temperature case it is in the class of problems for which Monte Carlo algorithms cannot be used in lattice simulations. Moreover, no viable alternative presently exists for doing such simulations. Thus any reliable results from the theory are extremely welcome. The problem is also interesting in light of the intense recent interest in QCD at finite baryon density [7].

The bound on the QCD free energy at fixed baryon chemical potential will play an essential role in bounding the nucleon mass and accordingly this problem will be treated first. This problem is rather unusual for a QCD inequality treatment. The approach is more commonly associated with bounding masses of particles via the study of correlation functions rather than with thermodynamically intensive quantities such as free energy densities. However, QCD inequalities have been used for intensive quantities in the past. Vafa and Witten [8] demonstrated that the vacuum energy for QCD including a θ term has an absolute minimum at $\theta = 0$. While the validity of a related argument by Vafa and Witten [8] that parity cannot be spontaneously broken has recently been questioned [9], the demonstration that the minimum vacuum energy is at $\theta = 0$ is clearly correct.

The Vafa-Witten proof is very simple. The integrand of the functional integral is given by $\prod_{i=\text{flavors}} \det(\not{D} + m_i) e^{-S_{YM} + i\theta\nu}$ where $\det(\not{D} + m_i)$, the functional determinant for a given flavor is real and non-negative [3], S_{YM} is the Yang-Mills action, and ν is the winding number. Thus, the only effect of setting θ to be nonzero is to multiply the rest of the integrand (which is real and positive) by a pure phase factor $e^{i\theta\nu}$. Since the real part of this phase factor is always less than or equal to unity (and the imaginary part will integrate to zero) one immediately deduces that the functional integral for nonzero θ is bounded from above by the integral with $\theta = 0$. Finally, identifying the functional integral as the generating function $Z(\theta) = e^{-V\mathcal{E}(\theta)}$, where V is the four dimensional volume and $\mathcal{E}(\theta)$ is the vacuum energy as a function of θ , one sees that the inequality for the generating function implies that $\mathcal{E}(\theta) > \mathcal{E}(0)$.

Here an argument analogous to that of Vafa and Witten will be given for the problem of two flavor QCD with degenerate quark masses at a nonzero chemical potential. The free energy density for QCD at fixed temperature and baryon chemical potential, $\mathcal{G}_B(T, \mu_B)$ is given in terms of the grand partition function $Z_B(T, \mu_B)$ as $\mathcal{G}_B(T, \mu_B) = -(\beta V_3)^{-1} \log[Z_B(T, \mu_B)]$ where V_3 is the three-dimensional volume of the system and β is the inverse temperature. In QCD with two degenerate flavors $Z_B(T, \mu_B)$ can be represented as a functional integral,

$$Z_B(T, \mu_B) = \int d[A] \left[\det\left(\not{D} + m - \frac{\mu_B}{N_c} \gamma_0\right) \right]^2 e^{-S_{YM}}, \quad (1)$$

where N_c is the number of colors (3 for the physical world), the functional determinant is over a single quark flavor, and the temperature is implemented by imposing periodic boundary conditions in time for the gluon fields $A(t + \beta) = A(t)$ with $\beta = 1/T$; similarly antiperiodic boundary conditions for the fermions are imposed in the functional determinant. The factor of $1/N_c$ simply reflects the fact that the chemical potential is for a baryon number and the baryon number of a single quark is $1/N_c$.

The difficulty in simulating this functional integral on the lattice stems from the fact that the functional determinant is not generally real and positive. From the perspective of QCD inequalities, however, this is not a bug, but a feature; it allows one to bound the partition function from above:

$$Z_B(T, \mu_B) \leq \int d[A] \left| \det\left(\not{D} + m - \frac{\mu_B}{N_c} \gamma_0\right) \right|^2 e^{-S_{YM}}. \quad (2)$$

In order for inequality (2) to be useful, its right-hand side needs to be expressed in terms of a physical quantity. Fortunately, it can be related to the free energy density of QCD with an *isospin* chemical potential [10]. An isospin chemical potential term is of the form $\mu_I \bar{q} \gamma_0 (\tau_3/2) q$, which implies that the functional integral for the grand partition function $Z_I(T, \mu_I) = \exp[-\beta V_3 \mathcal{G}_I(T, \mu_I)]$ is given by

$$Z_I(T, \mu_I) = \int d[A] e^{-S_{YM}} \times \det\left(\not{D} + m - \frac{\mu_I}{2} \gamma_0\right) \det\left(\not{D} + m + \frac{\mu_I}{2} \gamma_0\right), \quad (3)$$

where the two functional determinants are for the two flavors of quark and the opposite signs of the μ_I terms reflect the opposite values of I_3 for the two flavors. To proceed we use the fact that

$$\gamma_5 \left(\not{D} + m + \frac{\mu_I}{2} \gamma_0 \right) \gamma_5 = \left(-\not{D} + m - \frac{\mu_I}{2} \gamma_0 \right) = \left(\not{D} + m - \frac{\mu_I}{2} \gamma_0 \right)^\dagger, \quad (4)$$

where the last equality exploits the fact that in Euclidean space \not{D} is anti-Hermitian while the other two operators are Hermitian. Exploiting the cyclic property of the determinant allows one to write the second functional determinant in Eq. (3) as $\det[\not{D} + m + (\mu_I/2) \gamma_0] = \det\{\gamma_5 [\not{D} + m + (\mu_I/2) \gamma_0] \gamma_5\}$ and using Eq. (4) then gives

$$\det\left(\not{D} + m + \frac{\mu_I}{2} \gamma_0\right) = \left[\det\left(\not{D} + m - \frac{\mu_I}{2} \gamma_0\right) \right]^*. \quad (5)$$

Combining Eqs. (3) and (5) yields

$$Z_I(T, \mu_I) = \int d[A] \left| \det\left(\not{D} + m - \frac{\mu_I}{2} \gamma_0\right) \right|^2 e^{-S_{YM}}. \quad (6)$$

Finally, inequality (2) together with Eq. (6) implies that $Z_I[T, (2\mu_B/N_c)] \geq Z_B(T, \mu_B)$. This relation together with the definition of the free energy requires that

$$\mathcal{G}_B(T, \mu_B) \geq \mathcal{G}_I\left(T, \frac{2\mu_B}{N_c}\right), \quad (7)$$

which is the first principal result of this Letter.

Although this result was derived for two flavor QCD, the argument goes through for QCD with two degenerate light flavors and additional heavy flavors, provided the chemical potential term is understood as being the chemical potential for the up and down quark contributions to the baryon number rather than the full baryon chemical potential. The only change in the argument needed for this more general case is to include the functional determinant for the heavy flavors in all of the functional integrals. As the chemical potential does not apply to these heavy flavors, the functional determinants are the same as at $\mu = 0$ and hence are real and non-negative. Thus, they do not alter the preceding inequalities. This more general case is significant as in nature QCD has two light quarks which are nearly degenerate and additional heavy flavors. The result also applies in the general case to the full baryon chemical potential if one is in a regime in which the $\bar{s}\gamma_0 s = \bar{c}\gamma_0 c = \bar{b}\gamma_0 b = \bar{t}\gamma_0 t = 0$, since in this regime the total baryon number comes from up and down quarks. Such a regime occurs at zero temperature provided the chemical potential is below the critical chemical potential for strangeness condensation to occur.

A bound on the nucleon mass may be derived from inequality (7) using thermodynamic arguments. The bound applies to QCD with two degenerate light flavors and any number of heavy flavors. To begin, note that inequality (7) holds at all temperatures, including $T = 0$. At zero temperature there are no thermal fluctuations and the system is in a single quantum state which minimizes the free energy $G = H - \mu N$, where μ is the relevant chemical potential (either isospin or baryon) and $N = V_3 \rho$ is the related particle number. The role of the chemical potential at zero temperature is simply to alter the relative free energies of the various quantum states. Thus increasing the chemical potential from zero at $T = 0$ will have no effect until it is large enough so that the free energy of some other quantum state drops enough to equal that of the true vacuum. Accordingly, there is a critical value for the absolute value of the chemical potential at $T = 0$ below which the density is zero and the free energy is that of the vacuum state (which is conventionally taken to be zero). The critical chemical potentials are thus defined as follows:

$$\begin{aligned} \mathcal{G}_B(T=0, \mu_B) &= 0 & \text{for } |\mu_B| < \mu_B^c, \\ \mathcal{G}_B(T=0, \mu_B) &< 0 & \text{for } |\mu_B| > \mu_B^c, \\ \mathcal{G}_I(T=0, \mu_I) &= 0 & \text{for } |\mu_I| < \mu_I^c, \\ \mathcal{G}_I(T=0, \mu_I) &< 0 & \text{for } |\mu_I| > \mu_I^c. \end{aligned} \quad (8)$$

Inequalities (7) and (8), together imply the relation

$$\mu_B^c \geq \frac{N_c \mu_I^c}{2}. \quad (9)$$

It is straightforward to bound the μ_B^c from above by the nucleon mass using a simple variational argument.

Consider the quantum state of a single nucleon at rest. This state has energy M_N and baryon number unity; its free energy is $G_B = M_N - \mu_B$. Clearly this is less than or equal to zero for $\mu_B \geq M_N$. Thus, there is at least one state lower than the vacuum whenever $\mu_B \geq M_N$. This implies that $\mu_B^c \leq M_N$. This last inequality would become an equality if the system for μ_B just above its critical value formed an arbitrarily low density gas of nucleons (implying a second order transition). However, this is not what actually happens. Based on the extrapolation of finite nuclei densities and masses to infinite nuclear matter [11] one has a solid empirical basis to conclude that the transition is first order: for μ_B just above μ_B^c the system has nonzero energy and nonzero density. Thus $\mu_B^c = M_N - B$ where B is the binding energy per nucleon of infinite nuclear matter and the inequality, $\mu_B \geq M_N$, is not saturated. In any event, the inequality $\mu_B^c \leq M_N$ together with inequality (9) yields a bound on the nucleon mass.

$$M_N \geq \frac{N_c \mu_I^c}{2}. \quad (10)$$

For inequality (10) to be useful we need to know μ_I^c . For sufficiently small values of the quark mass we know that chiral perturbation theory accurately describes low energy excitations of the QCD vacuum. In chiral perturbation theory, the phase transition associated with increasing μ_I is second order and amounts to pion condensation, which implies $\mu_I^c = m_\pi$ [12]. More generally, μ_I^c corresponds to the state in QCD with the lowest energy per unit isospin. Let us denote such a state as X . There are two possibilities for X . If the transition is second order (as it is in chiral perturbation theory), then X is a single particle state; such a state is clearly at zero momentum so the energy can be identified as the mass of the particle. In this case, we can denote the mass of the state X , as m_X , and its isospin as I_X . The inequality (10) becomes

$$M_N \geq \frac{N_c m_X}{2 I_X}. \quad (11)$$

Although it is strongly believed that in nature the transition is second order, we do not require this in obtaining an inequality. Assume for the moment that the transition were first order. In this case the state of minimum energy per isospin would in fact be infinite isospin matter (in analogy to infinite nuclear matter), i.e., a state of uniform isospin density and uniform energy density filling all space. However, if this were the case, one could construct single particle states by taking large chunks of isospin matter (in analogy to large nuclei in a world where the electromagnetic interaction was shut off). The energy of such a state will have a bulk contribution scaling with the volume of the chunk and equaling $\mu_I^c I_3$ where I_3 is the isospin of the state. Corrections to this bulk value scale as the surface area and hence go like

$I_3^{2/3}$. Thus, by making I_3 arbitrarily large but finite one can create single particle states whose energy/isospin is arbitrary close μ_f^c . Identifying such a state as X one finds that there exists an X for which m_X/I_X can be made arbitrarily close to μ_f^c . Accordingly, regardless of whether the transition is first or second order one can conclude that inequality (10) implies there exists in QCD some single particle state X such that inequality (11) holds.

Note, the state X cannot be the nucleon itself as this is inconsistent with the inequality above; hence, inequality (11) makes a nontrivial prediction.

A few comments on the significance of these results is in order. Consider the implications of inequality (7). As noted previously, QCD at finite baryon chemical potential cannot be simulated on the lattice using Monte Carlo methods. It is generally believed that asymptotically high densities can be treated analytically from QCD using weak coupling, but nonperturbative, techniques along the line of BCS theory [7,13]. However, it is also generally believed that these densities are extraordinarily high and the regime where these methods work is unlikely to be relevant in laboratory experiments or in stellar physics. Thus, studies of phenomenological significance have been based on various models [7,14]. Ideally, such models should be as constrained as possible from QCD. Inequality (7) provides one possible basis for such a constraint. While the left-hand side of the inequality will be given by the model, the right-hand side is tractable in lattice QCD. The key point which allows such lattice simulations was recognized several years ago [10] and is simply that the integrand in the functional integral for Z_I in Eq. (3) is positive definite and hence is amenable to lattice studies. Lattice studies have been done both in the quenched approximation [15] and including dynamical quarks [16]. It is probably sensible to consider the dynamical lattice studies as being rather preliminary as they have been done on small lattices. However, as the lattice calculations improve, they may provide a strong constraint on models of QCD at finite baryon chemical potential.

Next consider inequality (11). It is very similar to the bounds derived by Nussinov for a wide class of QCD-inspired potential models [2] and to the bound derived by Nussinov and Sathiapalan in large N_c QCD [6]. The present result suffers in comparison to these results in that instead of bounding the nucleon mass by $N_c m_\pi/2$ the bound is with respect to $M_N \geq (N_c m_X/2I_X)$ where X is the state in QCD with lowest mass per unit isospin. In practice this disadvantage is fairly small — it so happens that in QCD the state with the lowest mass per unit isospin is the pion. The present bound has a clear advantage over the results in Refs. [2,6] in that it holds for QCD itself and is not restricted to the large N_c limit. Similarly, the present bound is much stronger than Weingarten's original bound [3] in that it holds for QCD with two

degenerate light flavors and does not require the *ad hoc* assumptions about the quark propagator in the presence of a background field. In any event inequality (11) is true phenomenologically. There does exist a state in QCD for which $M_N > (3M_X/2I_X)$ namely the case where X is the pion. Overall, one should view any bound on the nucleon mass in terms of other physical observables directly from a first principles treatment of QCD as being highly nontrivial.

The author thanks S. Nussinov for introducing him to the subject of QCD inequalities, for numerous illuminating discussions on the topic over the years, and as a guide to the relevant literature. D. Dakin is gratefully acknowledged for suggesting a number of improvements to the manuscript. This work was supported by the United States Department of Energy through Grant No. DE-FGO2-93ER-40762.

*Electronic address: cohen@physics.umd.edu

- [1] The current state of the art of lattice QCD can be found in *Lattice 2001: Proceedings of the 19th International Symposium on Lattice Field Theory, Berlin, Germany, 2001* [Nucl. Phys. (Proc. Suppl.) **106** (2002)].
- [2] S. Nussinov Phys. Rev. Lett. **51**, 2081 (1983).
- [3] D. Weingarten Phys. Rev. Lett. **51**, 1830 (1983).
- [4] E. Witten, Phys. Rev. Lett. **51**, 2351 (1983).
- [5] S. Nussinov and M. Lambert, Phys. Rep. **362**, 193–301 (2002).
- [6] S. Nussinov and B. Sathiapalan, Nucl. Phys. **B168**, 285 (1985).
- [7] Reviews of various aspects QCD at finite density can be found in M. Alford, Annu. Rev. Nucl. Part. Sci. **51**, 131 (2001); K. Rajagopal and F. Wilczek, hep-ph/0011333, in *At the Frontier of Particle Physics/Handbook of QCD*, edited by M. Shifman (World Scientific, Singapore, 2001).
- [8] C. Vafa and E. Witten, Phys. Rev. Lett. **53**, 535 (1984).
- [9] V. Azcoiti and A. Galante, Phys. Rev. Lett. **83**, 1518 (1999); T.D. Cohen, Phys. Rev. D **64**, 047704 (2001); X. Ji, Phys. Lett. B **554**, 33 (2003).
- [10] M. Alford, A. Kapustin, and F. Wilczek, Phys. Rev. D **59**, 054502 (1999).
- [11] This topic is discussed in any standard nuclear physics textbook. See, for example, P.E. Hodson, E. Gadioli, and E. Gadioli Erba, *Introductory Nuclear Physics* (Clarendon Press, Oxford, 1997).
- [12] D.T. Son and M. A. Stephanov, Phys. Rev. Lett. **86**, 592 (2001).
- [13] D.T. Son, Phys. Rev. D **59**, 094019 (1999).
- [14] Early examples of such model calculations include, M. Alford, K. Rajagopal, and F. Wilczek, Phys. Lett. B **422**, 247 (1998); Nucl. Phys. **B537**, 443 (1999).
- [15] J. B. Kogut and D. K. Sinclair, Phys. Rev. D **66**, 014508 (2002).
- [16] J. B. Kogut and D. K. Sinclair, hep-lat/0209054; Phys. Rev. D **66**, 034505 (2002).