Critical Dynamics of Model C Resolved

R. Folk[1] and G. Moser[2]
[1]Institute for Theoretical Physics, University of Linz, Austria
[2]Institute for Physics and Biophysics, University of Salzburg, Austria
(Received 10 January 2003; published 15 July 2003)

We analyze the field theoretic functions of the dynamical model C in two-loop order. Our results correct long-standing errors in these functions published by several authors. We discuss, in particular, the fixed points for the ratio w^* of the two time scales involved, as well as their stability. The regions of the "phase diagram," whose axes are the spatial dimension d and number of order parameter components n, correspond to these fixed points; previous authors have found, in addition, an anomalous region in which the scaling properties were unclear. We show that such an anomalous region does not exist. There are only two regions: one with a finite fixed-point w^* where the dynamical exponent $z = 2 + \alpha/\nu$, and another where $w^* = 0$ and z is equal to the model A value. We show how the one-loop result is recovered from the two-loop result in the limit $\epsilon = 4 - d$ going to zero.

DOI: 10.1103/PhysRevLett.91.030601 PACS numbers: 05.70.Jk, 64.60.Ak, 64.60.Ht

In 1974 Halperin, Hohenberg, and Ma studied the effects of energy conservation on the critical behavior of the simple time-dependent Ginzburg-Landau model [1] (model A). They introduced a model [2] in which a conserved secondary density is coupled to the nonconserved order parameter (OP). The OP relaxes with the relaxation rate Γ and the conserved density diffuses with the diffusion rate λ. Their calculations to first order in $\epsilon = 4 - d$ led to suggest various "phase diagrams" in the two dimensional space whose axes are the spatial dimension d and number of the OP components n. The "phases" in this space were characterized by the values of the stable fixed point for the time scale ratio $w = \Gamma/\lambda$. Three regions were found: region I with the fixed point $w^* = 0$; region II with $0 < w^* = n/(2 - n) < \infty$; and the anomalous region III with $w^* = \infty$. While in region I the OP dynamical critical exponent takes its model A value, and in region II the value $z = 2 + \alpha/\nu$ (α being the static exponent of the specific heat and ν the exponent of the correlation length), in the anomalous region the value of z and the validity of dynamical scaling behavior remained unclear. Model C, as it has been named since then, was subsequently treated within the field theoretical version of renormalization group theory up to two-loop order by Brezin and De Dominicis [3] and Murata [4].

The results of Brezin and De Dominicis corroborated a scenario where the anomalous region exists, and they obtained from their calculations at small values of ϵ the boundary curve separating region III from region II, although the extension of the anomalous region to the whole phase space could not be given. Further peculiarities, namely, a nonuniformity, appeared in the two-loop calculation when taking the limits $\epsilon \to 0$ and $w^* \to \infty$, which was believed to be the fixed-point value. Murata then recognized that in the anomalous region the fixed-point value of w^* showed an essential singularity and that this property might, even in the anomalous region, restore scaling and make the critical exponent take the value $z = 2 + \alpha/\nu$. Halperin, Hohenberg, and Ma then compared their second and higher order results with results known at the time [5] and found agreement with the boundary curves of [3]. On the other hand, the peculiarities in the two-loop order and the results of [4] led them to the supposition that the anomalous region might not exist, but they were unable to draw a definite conclusion.

This was the situation until 2002, when we calculated the field theoretic functions for a much more complicated model—for the dynamics of He^3-He^4 mixtures at the superfluid transition—containing the model C functions for $n = 2$ as the limit of the properly reduced more complicated model [6]. It turns out that our results differ from those of [3]. A calculation of the critical dynamics of model C for arbitrary n [7] then showed that the two-loop calculation of [3] was incorrect for all n. The correct result, however, allows us now to perform a systematic analysis of the model C in two-loop order, by calculating the complete phase diagram and performing the limit to the one-loop results. This shows that no anomalous region III exists and that the standard systematic ϵ expansion breaks down for $2 < n < 4$. Previous results for the boundary curves at small ϵ were due to a mistaken application of the ϵ expansion, which cannot catch the essential singularity appearing in the fixed-point function $w^*(\epsilon, n)$. The boundary line previously found between regions I and II are correct.

Model C describes the dynamics of a system characterized by an n-component nonconserved order parameter $\vec{\phi}_0(x, t)$ and a conserved secondary scalar density $m_0(x, t)$. In the absence of any mode coupling, the critical dynamics of the order parameter is purely relaxational, while the dynamics of the secondary density is determined by a diffusion process. This leads to the dynamic equations

 0031-9007/03/91(3)/030601(4)$20.00

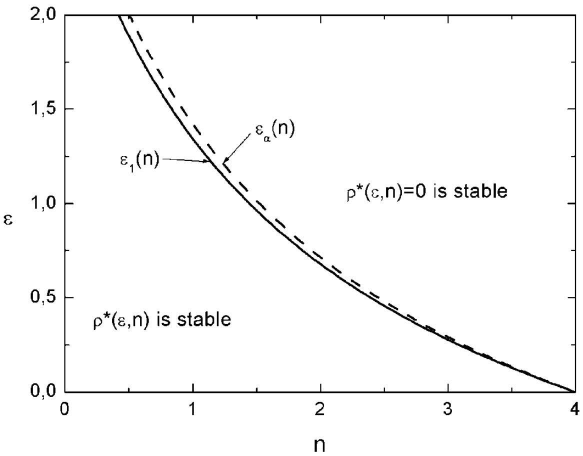

FIG. 1. Regions of existence of different fixed points: $\epsilon_\alpha(n)$ separates the region with nondiverging ($\gamma^* \equiv 0$, right side) from those with diverging ($\gamma^* \neq 0$, left side) specific heat (dashed curve). The solid curve $\epsilon_1(n)$ separates the region I where the fixed point $\rho^*(\epsilon, n) \equiv 0$ is stable (right side) from those where it is unstable (left side).

$$\frac{\partial \vec{\phi}_0}{\partial t} = -\overset{o}{\Gamma} \frac{\delta H}{\delta \vec{\phi}_0} + \vec{\theta}_\phi, \tag{1}$$

$$\frac{\partial m_0}{\partial t} = \overset{o}{\lambda} \nabla^2 \frac{\delta H}{\delta m_0} + \theta_m. \tag{2}$$

The kinetic coefficient of the order parameter $\overset{o}{\Gamma}$ and the Onsager coefficient $\overset{o}{\lambda}$ are related to the Gaussian-distributed stochastic forces $\vec{\theta}_\phi$ and θ_m by Einstein relations. The critical behavior of the thermodynamic derivatives follows from the static functional

$$H = \int d^d x \bigg\{ \frac{1}{2} \overset{o}{\tau} \vec{\phi}_0^2 + \frac{1}{2} \sum_{i=1}^{n} (\nabla \phi_{i0})^2 + \frac{\overset{o}{\tilde{u}}}{4!} (\vec{\phi}_0^2)^2 + \frac{1}{2} m_0^2 + \frac{1}{2} \overset{o}{\gamma} m_0 \vec{\phi}_0^2 - \overset{o}{h} m_0 \bigg\}. \tag{3}$$

Renormalizing the bare dynamical model equations leads to the renormalized counterparts of the bare quantities. The field theoretic functions describing the flow of the renormalized couplings and Onsager coefficients are obtained from the renormalization coefficients. The flow equations determine the fixed-point values and the universal asymptotic critical and subleading exponents.

If Γ sets the time scale of the dynamical model, the time scale ratio w of the renormalized Onsager coefficients

$$w = \frac{\Gamma}{\lambda}, \qquad \rho = \frac{w}{1 + w}, \tag{4}$$

and the suitable dynamical parameter ρ appear in the field theoretic functions. Restricting the static couplings to their *stable* static fixed-point values $u^* = \tilde{u}^* - 3\gamma^{*2}$ and γ^* for given ϵ and n, and using their ϵ expansion [8], we find the flow equation

$$l \frac{d\rho}{dl} = \beta_\rho(u^*, \gamma^*, \rho) = \rho(1 - \rho)\zeta_w(u^*, \gamma^*, \rho), \tag{5}$$

where

$$\zeta_w(u^*, \gamma^*, \rho) = \left(\rho - \frac{n}{2}\right)\gamma^{*2} - \frac{1}{2}\rho\gamma^{*2}\left\{\frac{n+2}{3}u^*(1 - L) + \rho\gamma^{*2}\left[\frac{n}{2} - \rho - \frac{(n+2)L}{2} + (1 + \rho)\ln(1 - \rho^2)\right]\right\} + \frac{n+2}{72}u^{*2}(2L - 1) \tag{6}$$

and $L = 3\ln(4/3)$. The solution of Eq. (5) for nonzero values of the flow parameter l describes the effective critical behavior of the dynamics. In the limit $l \to 0$, the time scale ratio reaches its fixed-point value and the critical exponents reach their universal asymptotic values. Within this asymptotic limit, scaling laws for the correlation functions can be derived.

Let us first summarize the results that are not in question (see Fig. 1). The solution $\rho^* = 0$ of the fixed-point equation $\beta(u^*, \gamma^*, \rho^*) = 0$, that exists in the whole phase space, is stable in the region I. This region is separated from the other regions by the line $c\eta = \alpha/\nu$ ($c = 2L - 1$) containing the static exponent η of the correlation function at the phase transition. c is the coefficient appearing in model A in which $z = 2 - c\eta$ is the dynamical critical exponent. This boundary curve extends the stability of $\rho^* = 0$ to the region where the stable fixed-point value of the static coupling γ^* is nonzero [3].

We are interested in the remaining region, and look for a nonzero fixed-point *function* $\rho^*(\epsilon, n) < 1$ at fixed ϵ. This is found by looking for zeros of the function ζ_w. Such a finite nonzero fixed-point function leads immediately to scaling, i.e., to the same scaling exponent α/ν [3] for the relaxation rate Γ and the diffusion constant λ.

The usual approach for finding the fixed-point function in the ϵ expansion is restricted to the region $n < 2$ since, at $n = 2$, the lowest order solution for $\rho^*(\epsilon, n) \to 1$ diverges due to the logarithmic term in the two-loop order, and this is clearly unphysical. However, it is not necessary to expand in ϵ: the nonlinear equation can be solved directly, and then the divergence does not appear. The solutions for the fixed-point function are shown in Fig. 2(a) for different ϵ. It is a smooth function starting at zero for $n = 0$, rises to a maximum, and decreases to zero at its *existence* boundary. This *existence* boundary coincides with the *stability* boundary of the fixed point $\rho^* = 0$. One also observes convergence to the one-loop result in the limit $\epsilon \to 0$ [9]. For very small values of ϵ, one might think that another region exists, where the

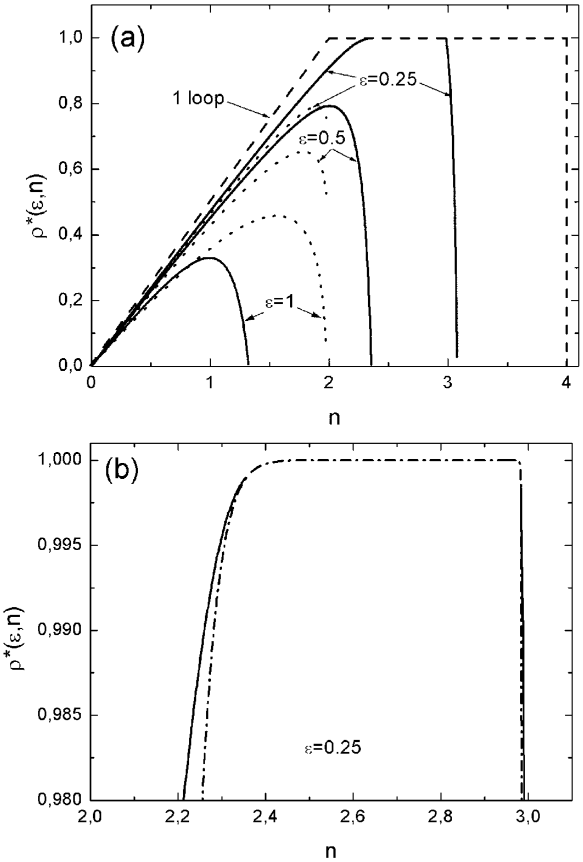

FIG. 2. (a) Fixed-point function $\rho^*(\epsilon, n)$ (n OP dimension) at different spatial dimensions d ($\epsilon = 4 - d$), solid curve numerical solution. The maximum of the function is always *smaller* than 1 [see Eq. (7)] but this is not visible for ϵ near zero. Dotted curves calculated from strict ϵ expansion. It breaks down at $n = 2$. (b) Enlargement of the region near $\rho^*(\epsilon, n) = 1$. Shown here is the merging of the numerical solution (solid curve) into the analytic solution Eq. (7) (dash-dotted curve) valid near $\rho^*(n) = 1$. Even at this resolution, one cannot see that the fixed-point function does not reach the value 1.

maximum of the fixed-point function remains constant at $\rho^* = 1$, corresponding to $w^* = \infty$. But this is not the case. Taking into account only the leading terms of Eq. (6), in this region one finds the behavior of the fixed-point function

$$\rho^*_{as}(\epsilon, n) = 1 - 0.5 \exp[-a(\epsilon, n)/\gamma^{*2}] \equiv 1 - x \qquad (7)$$

with

$$a(\epsilon, n) = \frac{n}{2} - 1 + \frac{n+2}{6} u^*(1 - L) + \frac{\gamma^{*2}}{2}\left[\frac{n}{2} - 1 - \frac{(n+2)L}{2}\right] - \frac{(n+2)u^{*2}}{72\gamma^{*2}}(2L - 1), \qquad (8)$$

proving to be smaller than 1. Figure 2(b) verifies the crossover of the numerical solution to the asymptotic solution Eq. (7), but even in this enlarged figure the finiteness of the time scale ratio w^* cannot be displayed. This feature persists in a self-similar way even for higher magnifications. As in one-loop order, a third fixed-point function $\rho^* \equiv 1$ exists in the entire phase space, but this function is unstable at higher than one-loop order (see below).

The next step is to consider the stability of the different fixed-point functions in the phase space. The fixed-point function is stable when the stability function

$$\begin{aligned}\omega_\rho(\epsilon, n) &\equiv \frac{\partial \beta_\rho}{\partial \rho}(u^*, \gamma^*, \rho^*) \\ &= (1 - 2\rho^*)\zeta_w(u^*, \gamma^*, \rho^*) \\ &\quad + \rho^*(1 - \rho^*)\frac{\partial \zeta_w}{\partial \rho}(u^*, \gamma^*, \rho^*)\end{aligned} \qquad (9)$$

with

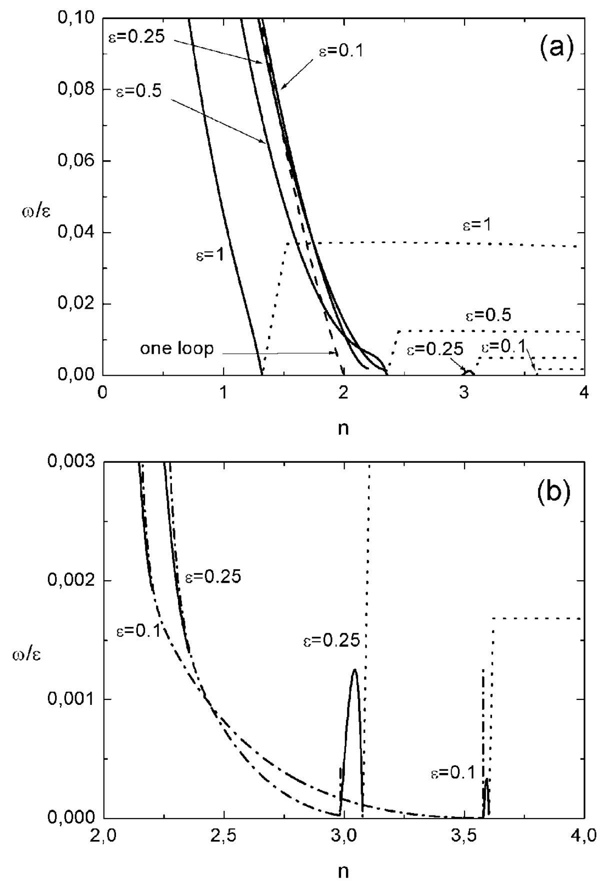

FIG. 3. (a) Dynamic stability function $\omega_\rho(\epsilon, n)$ Eq. (9) divided by ϵ for various ϵ. Solid curve, numerical solution; dashed curve ending at $n = 2$, one-loop result; dotted curve, stability function for $\rho \equiv 0$. A gap seems to exist between the solid and the dotted curves. (b) same as (a) but magnified. The dashed-dotted curve shows $\omega_\rho^{(as)}(\epsilon, n)$ Eq. (11). It connects the two parts of the numerical solution and is always larger than zero.

$$\frac{\partial \zeta_w}{\partial \rho}(u^*, \gamma^*, \rho) = \gamma^{*2}\left\{1 - \frac{n+2}{6}u^*\left(1 - L\right) - \rho\gamma^{*2}\left[\frac{n}{2} - \rho - \frac{(n+2)L}{2} + (1+\rho)\ln(1-\rho^2)\right] + \frac{1}{2}\rho^2\gamma^{*2}\left[\frac{1+\rho}{1-\rho} - \ln(1-\rho^2)\right]\right\} \quad (10)$$

is larger than zero. For $\rho^* \equiv 1$ to be stable, $\omega_\rho = -\zeta_w(u^*, \gamma^*, 1)$ has to be larger than zero, but the logarithmic term in Eq. (6) goes to $+\infty$, proving instability in the whole phase space. The stability of the fixed-point function $\rho^*(\epsilon, n)$, which is smaller than 1 everywhere, is governed by the derivative of the ζ_w function. We have plotted the corresponding stability function $\omega_w(\epsilon, n)$ divided by ϵ for different ϵ in Fig. 3(a). One observes that the two-loop stability function, for small ϵ, tends to the one-loop stability function; however, for small ϵ, there seems to be a gap between the value of n where the stability function goes to zero and the *existence* boundary of $\rho^*(\epsilon, n)$. This is just the region where $\rho^*(\epsilon, n)$ seems to be almost equal to 1.

Inserting the asymptotic fixed-point function ρ^*_{as} from Eq. (7) into $\omega_\rho(\epsilon, n)$ and taking leading terms, we get

$$\omega_\rho^{(as)}(\epsilon, n) = \gamma^{*4} + x\gamma^{*2}\left[1 - \frac{n+2}{6}u^*(1-L) - \gamma^{*2}\left(\frac{n}{2} - 1 - \frac{(n+2)L}{2} + \frac{5}{2}\ln 2x\right)\right]. \quad (11)$$

Thus we find the value of the stability function within this region to be larger than zero. It goes to zero only where its *stability* boundary coincides with the *existence* boundary of the fixed-point function [see Fig. 3(b)]. The peculiar behavior of the stability function is the result of the formation of the sharp edge in the fixed-point function near the *existence* border. The stability function, for $\rho^* \equiv 0$, has two kinds of behavior: within $\epsilon_1(n)$ and $\epsilon_\alpha(n)$, it is given by $c\eta - \alpha/\nu$, and beyond $\epsilon_\alpha(n)$, by $c\eta$. The one-loop values for the fixed points and stability exponents at $\epsilon = 1$ are vastly different in two-loop order: $w^*_{2\text{loop}}(1, 1) = 0.49$ instead of $w^*_{1\text{loop}}(1, 1) = 1$, $\omega_w^{(2\text{loop})}(1, 1) = 0.045$ instead of $\omega_w^{(1\text{loop})}(1, 1) = 0.17$.

Thus we find the following picture (see again Fig. 1). In the region right to the stability line $\epsilon_1(n)$ (solid line) strong scaling holds with both, the OP and m scaling with $z = 2 + \alpha/\nu$. In the region left to the stability line $\epsilon_\alpha(n)$ (dashed curve) the equations decouple. The OP scales with $z = 2 - c\eta$ and m with $z_m = 2$. In the region between the stability lines weak scaling holds, where the OP scales with $z = 2 - c\eta$ and m with $z_m = 2 + \alpha/\nu$.

In conclusion, we remark that for physical applications, such as nonequilibrium relaxation [10], finite size effects in the dynamics [11] of the Ising model, or systems with quenched impurities [12], the fixed-point value and the effective value of the time scale ratio w are of relevance. Similar peculiarities as in model C have been observed in the model describing the critical dynamics at a tricritical point [13]. These may be resolved in a similar way by considering the fixed point as a nonlinear solution of its β function. Extensive use is made of such a view in the interpretation of effective critical behavior. Finally, we mention that model C is contained in more complicated models for the superfluid critical dynamics in which mode coupling terms are present.

We acknowledge support from the Fonds zur Förderung der wissenschaftlichen Forschung (Project No. P15247).

[1] B. I. Halperin, P. C. Hohenberg, and Shang-keng Ma, Phys. Rev. Lett. **29**, 1548 (1972) (model *A*).
[2] B. I. Halperin, P. C. Hohenberg, and Shang-keng Ma, Phys. Rev. B **10**, 139 (1974) (model *C*).
[3] E. Brezin and C. De Dominicis, Phys. Rev. B **12**, 4954 (1975).
[4] K. K. Murata, Phys. Rev. B **13**, 2028 (1976).
[5] B. I. Halperin, P. C. Hohenberg, and Shang-keng Ma, Phys. Rev. B **13**, 4119 (1976).
[6] R. Folk and G. Moser, Phys. Rev. Lett. **89**, 125301 (2002).
[7] R. Folk and G. Moser, Acta Phys. Slovaca **52**, 285 (2002).
[8] In principle one might use the flow equation for the static coupling parameters. However, the fourth order coupling does not reach a real fixed-point in two-loop order and one has to rely on the Borel summed β functions, which are known for special values of n but not in the whole phase space. See R. Schloms and V. Dohm, Nucl. Phys. **B328**, 639 (1989).
[9] For $\epsilon \equiv 0$ one has to take into account that the static fixed point $u^* = 0$ has to be replaced by the solution of the static flow equation leading to a logarithmic dependence on l instead of a constant fixed-point value in the dynamic β function.
[10] K. Oerding and H. K. Janssen, J. Phys. A **26**, 3369 (1993).
[11] W. Koch and V. Dohm, Phys. Rev. E **58**, R1179 (1998).
[12] G. Grinstein, Shang-keng Ma, and G. F. Mazenko, Phys. Rev. B **15**, 258 (1977).
[13] E. D. Siggia and D. R. Nelson, Phys. Rev. B **15**, 1427 (1977).